MATH2020A Midterm Review

The midterm will cover everything that we did on 2D- and 3D-integration. It will not cover any vector analysis.

The main topics that we went through were the following:

- Riemann Sums and Integrable/Non-Integrable Functions;
- Fubini's Theorem and Special Domains in 2 and 3 Dimensions;
- 2-Dimensional Integration in Polar Coordinates;
- 3-Dimensional Integration in Cylindrical and Spherical Coordinates;
- General Changes of Variables.

We also touched on:

- Improper Integrals;
- Various Interpretations of Integration (Mass, Volume, Inertia, Center of Mass, etc.);

Any of these topics may appear on the midterm.

1 Riemann Sums

Recall that we defined integration as the limit of Riemann sums. This idea still holds in any dimension (for us we will consider the 2- and 3-dimensional cases).

To take a Riemann sum of a function f over a rectangle $R \subseteq \mathbb{R}^2$, we need a partition P to subdivide R into smaller sub-rectangles $R_{k,l}$ and also choices of points $(x_{k,l}, y_{k,l})$ in each of those sub-rectangles.

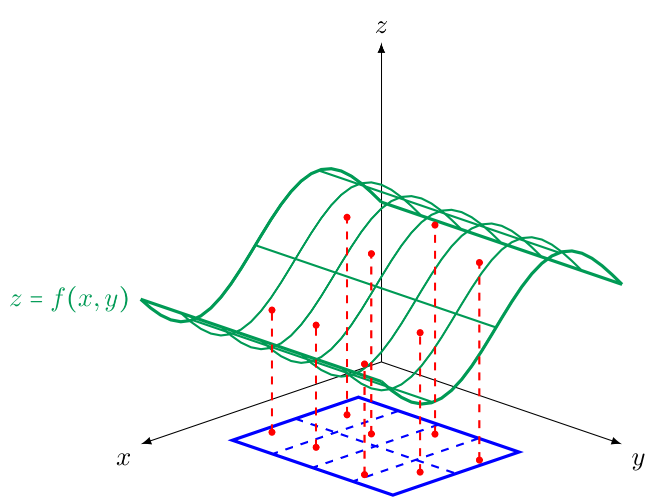


Figure 1: In each sub-rectangle $R_{k,l}$ of a partition P, we choose a point $(x_{k,l}, y_{k,l})$ and take the value of the function f there.

The associated Riemann sum is then

$$S(P,f,x_{k,l},y_{k,l}) = \sum_{k=1}^{m}\sum_{l=1}^{n} f(x_{k,l},y_{k,l})\cdot \Delta A_{k,l}. \tag{1}$$

This sum looks at each sub-rectangle $R_{k,l}$, takes the value of the function f at the chosen point $(x_{k,l},y_{k,l})$ and multiplies it by the area $\Delta A_{k,l}$ of $R_{k,l}$.

When the partitions get finer and finer, we get a better approximation of the (signed) volume under the graph. If the limit exists as we take $\|P\|\to 0$, then we say the function is integrable, that is

$$\iint_R f(x,y)\,dA = \lim_{\|P\|\to 0} S(P,f,x_{k,l},y_{k,l}). \tag{2}$$

To do this in generality, we need to show that this is independent of the choices of partitions and points. Essentially this means that whatever we choose does not affect the result.

In practice, to get an idea of the integral, we look at convenient families of partitions and points, where we can tune the size without much issue. These usually take an interval $[a,b]$ and split it into m equally sized parts which has the nice formula:

$$P: a = a + \frac{0(b-a)}{m} = s_0 < a + \frac{b-a}{m} = s_1 < \ldots < a + \frac{k(b-a)}{m} = s_k < \ldots < b = a + \frac{m(b-a)}{m} = s_m. \tag{3}$$

The associated points are usually picked to be one of the end points of each sub-interval.

The 3-dimensional case is similar, except we have 3 coordinates instead of 2 so we divide boxes into sub-boxes with partitions.

Example 1. Using Riemann sums, compute the value of the integral of the function $f(x,y) = x + 2y^2$ over the rectangle $R = [0,1]\times[1,3]$.

Solution. We have lots of choice in choosing partitions. To make things easier, we choose simple ones that divide the region into evenly spaced sub-rectangles. We take

$$Q_m : 0 = s_0 < \frac{1}{m} = s_1 < \ldots < \frac{k}{m} = s_k < \ldots < 1 = s_m \tag{4}$$

and

$$R_n : 1 = t_0 < 1 + \frac{2}{n} = t_1 < \ldots < 1 + \frac{2l}{n} = t_l < \ldots < 3 = t_n. \tag{5}$$

To simplify things, we also choose the "top-right" corner of each sub-rectangle. That is, in each sub-rectangle $R_{k,l} = [s_{k-1},s_k]\times[t_{l-1},t_l] = \left[\frac{k-1}{m},\frac{k}{m}\right]\times\left[1+\frac{2(l-1)}{n},1+\frac{2l}{n}\right]$, we choose $(x_{k,l},y_{k,l}) = \left(\frac{k}{m},1+\frac{2l}{n}\right)$.

The Riemann sum here is

$$\begin{aligned}
S(P_{m,n},f,x_{k,l},y_{k,l}) &= \sum_{k=1}^{m}\sum_{l=1}^{n} f(x_{k,l},y_{k,l})\cdot\Delta A_{k,l} \\
&= \sum_{k=1}^{m}\sum_{l=1}^{n}\left(\frac{k}{m} + 2\left[1+\frac{2l}{n}\right]^2\right)\cdot\frac{1}{m}\cdot\frac{2}{n} \\
&= \sum_{k=1}^{m}\sum_{l=1}^{n}\left(\frac{2k}{m^2 n} + \frac{4}{mn} + \frac{16l}{mn^2} + \frac{16l^2}{mn^3}\right) \\
&= \frac{m(m+1)n}{m^2 n} + \frac{4mn}{mn} + \frac{8mn(n+1)}{mn^2} + \frac{8mn(n+1)(2n+1)}{3mn^3} \\
&= \frac{m+1}{m} + 4 + \frac{8(n+1)}{n} + \frac{8(n+1)(2n+1)}{3n^2} \\
&\longrightarrow_{m,n\to\infty} 1 + 4 + 8 + \frac{16}{3} \\
&= \boxed{\frac{55}{3}.}
\end{aligned}$$

□

2 Fubini's Theorem and Special Domains

Instead of using Riemann sums, we have various "shortcuts" that we regularly use to evaluate 2D- and 3D-integrals. One particularly helpful thing to do is to identify the domain of integration. It is generally easiest if we can describe it as a special kind of domain.

In 2 dimensions, we try to write domains as

$$\{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, \quad g_1(x) \le y \le g_2(x)\} \tag{6}$$

or

$$\{(x,y) \in \mathbb{R}^2 \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}. \tag{7}$$

In 3 dimensions, special domains are of the form

$$\{(x,y,z) \in \mathbb{R}^3 \mid a \le x \le b, \quad g_1(x) \le y \le g_2(x), \quad u_1(x,y) \le z \le u_2(x,y)\}. \tag{8}$$

There are five other types of special domains where we permute the roles of x, y, and z.

Essentially, one variable has static bounds, the next has bounds based on the first one, and the next has bounds based on the other two.

With these special domains in mind, we can use Fubini's Theorem to evaluate integrals.

Theorem 2 (Fubini's Theorem (2D Version))**.** *Let $f(x,y)$ be a continuous function on a closed (and bounded) domain D.*

- *If D is of the form*

$$\{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, \quad g_1(x) \le y \le g_2(x)\}, \tag{9}$$

  then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx; \tag{10}$$

- *If D is of the form*

$$\{(x,y) \in \mathbb{R}^2 \mid c \le y \le d, \quad h_1(y) \le x \le h_2(y)\}, \tag{11}$$

  then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy. \tag{12}$$

Theorem 3 (Fubini's Theorem (3D Version))**.** *Let $f(x,y)$ be a continuous function on a closed (and bounded) domain D. If D is of the form*

$$\{(x,y,z) \in \mathbb{R}^3 \mid a \le x \le b, \quad g_1(x) \le y \le g_2(x), \quad u_1(x,y) \le z \le u_2(x,y)\}, \tag{13}$$

then

$$\iiint_D f(x,y,z)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx. \tag{14}$$

Note that in the 3D case, there are five other versions of this, where the roles of x, y, and z are permuted.

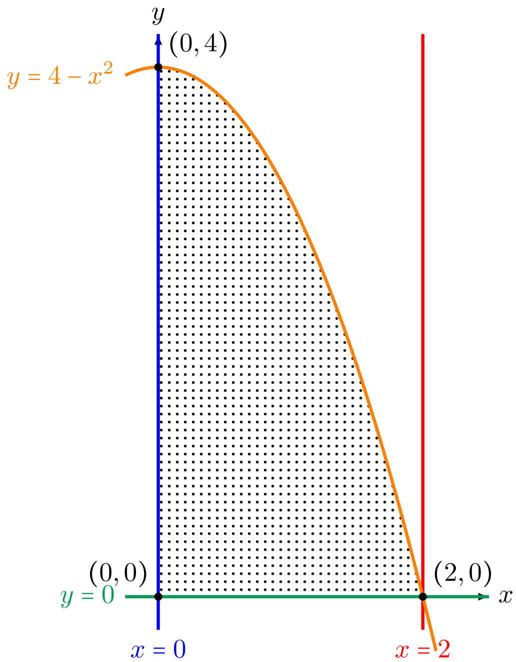
An important use of Fubini's Theorem is swapping the order of integration. This requires us to think of our domain as a special domain in multiple ways. In theory, integrating in either order does not change the value of the result, however, this can have consequences in practice since some integrals are difficult or do not have closed forms.

Example 4. Integrate

$$\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx. \tag{15}$$

Solution. At first glance, this is not a function of y that we readily know the anti-derivative of. Some useful thoughts are that e^{2y} is a "nice" function that we recognize and so is $\frac{1}{4-y}$ and so integration by parts might be possible here. (Some further thought might show that integration by parts would not be particularly helpful since integrating $\frac{1}{4-y}$ gives $-\ln(4-y)$ and differentiating $\frac{1}{4-y}$ gives $\frac{1}{(4-y)^2}$. Neither of which combine nicely with e^{2y}.)

In general, it is usually helpful to look at the domain of integration and see if swapping the order of integration can simplify things.



One thing to be careful about here is that the integrand

$$f(x,y) = \frac{xe^{2y}}{4-y} \tag{16}$$

is actually not defined at the boundary point $(0,4)$, however as we approach that point, we have both $x \to 0$ and $y \to 4$ and these cancel out in the limit. This allows Fubini's Theorem to apply.

We can rewrite the section of the boundary curve

$$y = 4 - x^2 \longleftrightarrow x = \sqrt{4-y}. \tag{17}$$

Swapping the order of integration, we get

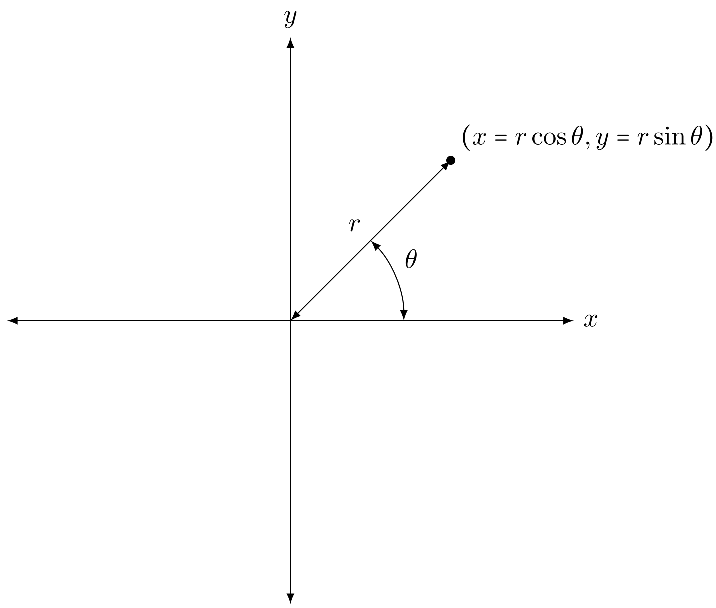
$$\begin{aligned}\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx &= \int_0^4 \int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\,dx\,dy \\ &= \int_0^4 \frac{4-y}{2}\frac{e^{2y}}{4-y}\,dy \\ &= \int_0^4 \frac{1}{2}e^{2y}\,dy \\ &= \boxed{\frac{1}{4}(e^8-1).}\end{aligned}$$

□

3 Polar Coordinates in 2 Dimensions

One useful thing in 2 dimensions is to convert between polar coordinates and Cartesian coordinates. This is given by the change of coordinates

$$\begin{cases} x &= r\cos\theta, \\ y &= r\sin\theta. \end{cases} \tag{18}$$



In general, polar coordinates are useful when dealing with circular arcs or radial lines since these are described by constant r or constant θ. (Keep in mind that static bounds are generally the easiest things to integrate.)

When changing from Cartesian to polar coordinates we need to keep in mind that the area element changes:

$$dx\,dy \longleftrightarrow r\,dr\,d\theta. \tag{19}$$

Example 5. Evaluate

$$\int_{-\frac{3}{2}}^{0}\int_{-\sqrt{3}x}^{\sqrt{9-x^2}}(x^2+y^2)\,dy\,dx + \int_{0}^{\frac{3\sqrt{3}}{2}}\int_{\frac{1}{\sqrt{3}}x}^{\sqrt{9-x^2}}(x^2+y^2)\,dy\,dx \tag{20}$$

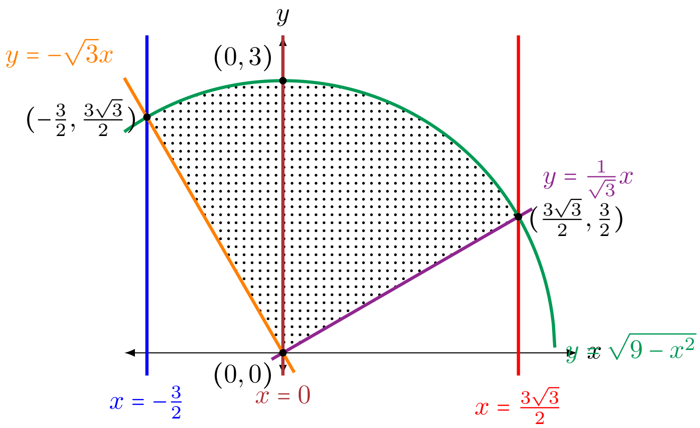
Solution. We have two integrals here with the same integrand. This likely means that the combined domain is not convenient in Cartesian coordinates. We also see that the circular arc $y = \sqrt{9 = x^2}$ is involved, yet again implying that polar coordinates is the way to go here. Sketching the domain out, we see the following:

Thus, the combined domain is a circular segment. We can see that the boundary line segments correspond to:

$$\begin{cases} y = \sqrt{9-x^2} & \longleftrightarrow r = 3, \\ y = -\sqrt{3}x & \longleftrightarrow \theta = \dfrac{2\pi}{3}, \\ y = \dfrac{1}{\sqrt{3}}x & \longleftrightarrow \theta = \dfrac{\pi}{6}. \end{cases} \tag{21}$$

Also the integrand is

$$x^2 + y^2 \longrightarrow r^2. \tag{22}$$



Altogether, we get

$$\begin{aligned}\int_{-\frac{3}{2}}^{0}\int_{-\sqrt{3}x}^{\sqrt{9-x^2}}(x^2+y^2)\,dy\,dx + \int_{0}^{\frac{3\sqrt{3}}{2}}\int_{\frac{1}{\sqrt{3}}x}^{\sqrt{9-x^2}}(x^2+y^2)\,dy\,dx &= \int_{\frac{\pi}{6}}^{\frac{2\pi}{3}}\int_{0}^{3} r^2\cdot r\,dr\,d\theta \\ &= \left[\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} d\theta\right]\cdot\left[\int_{0}^{3} r^3\,dr\right] \\ &= \frac{\pi}{2}\cdot\frac{81}{4} \\ &= \boxed{\frac{81\pi}{8}.}\end{aligned}$$

□

4 Cylindrical Coordinates and Spherical Coordinates in 3 Dimensions

In 3 dimensions, we also have some common coordinate systems given by cylindrical coordinates and spherical coordinates.

Cylindrical coordinates essentially uses polar coordinates for x and y, while leaving the z coordinate unchanged.

$$\begin{cases} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{cases} \tag{23}$$

The area element here changes by

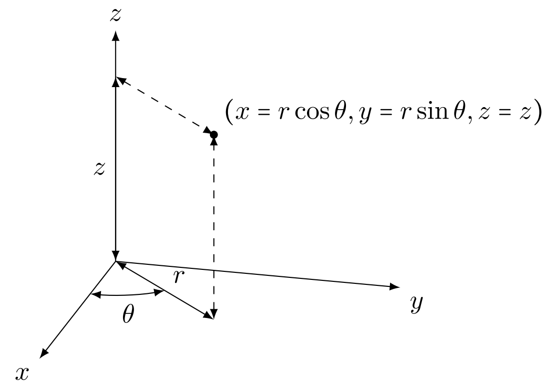
$$dx\,dy\,dz \longleftrightarrow r\,dr\,d\theta\,dz. \tag{24}$$

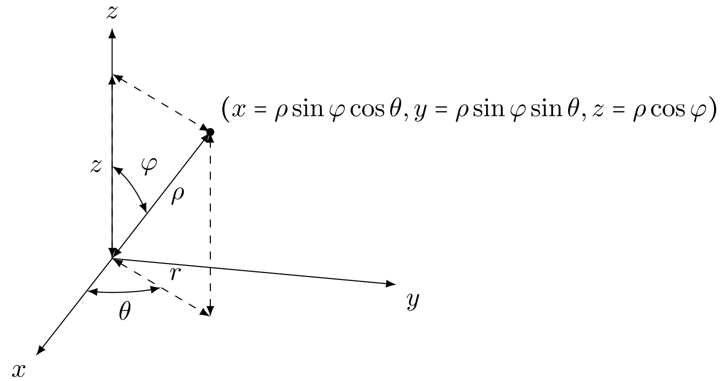
Spherical coordinates are given by

$$\begin{cases} x &= \rho\sin\varphi\cos\theta, \\ y &= \rho\sin\varphi\sin\theta, \\ z &= \rho\cos\varphi. \end{cases} \tag{25}$$

For spherical coordinates, the area element here changes by

$$dx\,dy\,dz \longleftrightarrow \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta. \tag{26}$$





The big thing here is deciding which coordinate system to use. While convenient, it is always important to ask whether you can write bounds for either r or ρ in a easy to compute manner. If not, you should probably rethink the choice of coordinate system. It also helps to think about how the angles sweep out the solid that we are integrating over.

Example 6. Find the volume of the solid created bound the cylinder over the circle centered at the origin on the xy-plane with radius $\sqrt{5}$ and the sphere of radius 9 also centered at the origin.

Solution. Even though this set up involves a sphere, it is easier to use cylindrical coordinates since the boundary faces are cylindrical (except for the caps on the top and bottom).

We see that the volume is given by the integral

$$\begin{aligned}\int_0^{2\pi}\int_0^{\sqrt{5}}\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta &= \left[\int_0^{2\pi} d\theta\right]\cdot\left[\int_0^{\sqrt{5}} 2r\sqrt{9-r^2}\,dr\right]\\ &= 2\pi\cdot\left[-\frac{2}{3}(9-r^2)^{\frac{3}{2}}\Big|_{r=0}^{r=\frac{1}{4}}\right]\\ &= 2\pi\cdot\left(18-\frac{16}{3}\right)\\ &= \boxed{\frac{76\pi}{3}.}\end{aligned}$$

□

5 Changes of Coordinates

Sometimes, it is convenient to use a different coordinate system than those considered above. If we have a change of coordinates

$$\begin{cases} x &= g(u,v), \\ y &= h(u,v), \end{cases} \tag{27}$$

then the area element will change with respect to the Jacobian determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{vmatrix}. \tag{28}$$

Note that this may be a function of u and v, and so areas will distort differently at different points.

In essence, we have the following

$$dx\,dy \longleftrightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv. \tag{29}$$

Similar formulae hold for 3-dimensional changes of coordinates.

As with any coordinate change, we need to transform the bounds of integration from xy-coordinates to uv-coordinates and vice versa. Drawing figures can help in visualizing how these map to one another.

Example 7. Consider the change of coordinates

$$\begin{cases} u &= 2x + y, \\ v &= x - y. \end{cases} \tag{30}$$

Invert the system and find the Jacobian $\frac{\partial(x,y)}{\partial(u,v)}$. How does the triangle bounded by $y = 0$, $y = x$, and $x + 2y = 2$ change under this transformation?

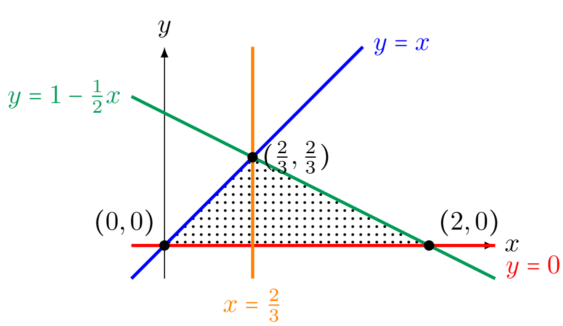
Solution. We see that

$$\begin{cases} x &= \dfrac{1}{3}u + \dfrac{1}{3}v, \\ y &= \dfrac{1}{3}u - \dfrac{2}{3}v. \end{cases} \tag{31}$$

This means that the Jacobian determinant is

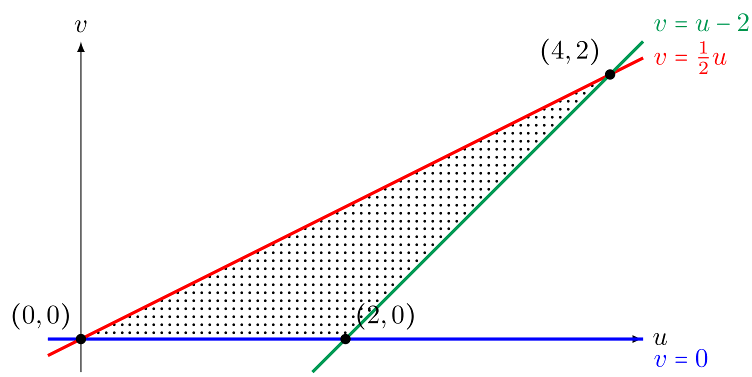
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{vmatrix} = -\frac{1}{3}. \tag{32}$$

By sketching the domain of integration, we have



We can see that the boundary curves correspond via

$$\begin{cases} y = x & \longleftrightarrow v = 0, \\ y = 0 & \longleftrightarrow v = \frac{1}{2}u, \\ y = 1 - \frac{1}{2}x & \longleftrightarrow \frac{1}{3}u - \frac{2}{3}v = 1 - \frac{1}{6}u - \frac{1}{6}v \longleftrightarrow v = u - 2. \end{cases} \tag{33}$$



□

One thing to see here is that the determinant of the Jacobian was negative. This corresponds to the relative positions of the red, blue, and green lines swapping.

6 Improper Integrals

When dealing with improper integrals in 2 and 3 dimensions, it helps to think about what the area of integration is. Like in 1 dimension, there are two kinds of improper integral:

- ones with unbounded domain;
- ones with unbounded integrand.

In either case, try to come up with recognizable shapes dependent on a parameter that slowly approach the desired domain.

Example 8. Evaluate

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(x^2+1)(y^2+1)}\,dy\,dx. \tag{34}$$

Solution. The domain of integration here is the entire plane. We can approximate this in several ways. One way is to use circles of radius R and take the limit as $R \to \infty$. This would work, but the integrand is not nicely suited for polar coordinates. In this case, we can use boxes of the form $[-k, k] \times [-k, k]$ and take the limit as $k \to \infty$.

We write

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{(x^2+1)(y^2+1)}\,dy\,dx &= \lim_{k\to\infty}\int_{-k}^{k}\int_{-k}^{k}\frac{1}{(x^2+1)(y^2+1)}\,dy\,dx \\ &= \lim_{k\to\infty}\int_{-k}^{k}\frac{\arctan(k)-\arctan(-k)}{x^2+1}\,dx \\ &= \lim_{k\to\infty}\big(\arctan(k)-\arctan(-k)\big)^2 \\ &= \pi^2.\end{aligned}$$

□

7 Interpretations of Integration

Over the course so far, we have seen many ways to think of integration as telling something about a function or a domain. There is nothing particularly special here except to remember which formulae correspond to what.

When dealing with physical quantities, it sometimes helps to think in terms of units and dimensions. Integration in 2 dimensions corresponds to area ($[L]^2$), while 3 dimensions corresponds to volume ($[L]^2$). This is consistent with units for quantities such as density ($[M] \cdot [L]^{-3}$) and mass ($[M]$).