MATH2020A Lecture 9 Notes

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Last time, we began a (rather involved) example about a change of coordinates in 3-dimensions. Remember that integration is "more of an art than a science" – there are many tools at your disposal to try to solve them and most of them work, you just want to minimize your effort.

In our case, we saw that the region of integration was an octahedron. Without many symmetries available (due to the integrand), we cannot reduce the problem to the first octant, where the bounds of integration are convenient. As such, we would probably have to split the region into 4 or more pieces and integrate over them.

Alternatively, what we ended up doing, was use a change of coordinates to map most of these boundary faces to the faces of a cube, which is easier to integrate over. We had some extra work since we do not get the whole cube, as two corners have been sliced off. Ultimately, we can see that its easier to integrate over the whole cube and then subtract the pieces that are not actually there, since those regions are nicely described.

Example 1. Let

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid |x| + |y| + |z| \leqslant 1\}. \tag{1}$$

Evaluate

$$\iiint_D (x + y + z)^4 \, dV. \tag{2}$$

Remark 2. Before we begin, we first note that we can use symmetries to make things slightly easier. We have to be careful since we have to make sure that the integrand does not change under the symmetries we use.

For example, consider the symmetry $(x, y, z) \longleftrightarrow (-x, -y, -z)$. We can check that

$$\big((-x) + (-y) + (-z)\big)^4 = \big(-(x + y + z)\big)^4 = (x + y + z)^4 \tag{3}$$

and so the integrand respects this symmetry.

On the other hand, consider $(x, y, z) \longleftrightarrow (x, y, -z)$. We compute that

$$\big(x + y + (-z)\big)^4 \neq (x + y + z)^4 \tag{4}$$

in general and so we cannot reduce using this symmetry.

What this means here is that we cannot use the first octant trick from our earlier examples since we need the integrand to satisfy all eight symmetries given by reflecting over coordinate planes.

Solution. Let us first try to get an idea of the shape we are integrating over. Setting $z = 0$, we have $|x| + |y| \leqslant 1$ which looks like a (rotated) square.

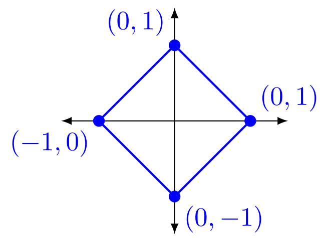

Figure 1: Projecting the domain D onto the xy-plane.

The boundary lines here are

$$\begin{cases} x+y=1, \\ x+y=-1, \\ x-y=1, \\ x-y=-1. \end{cases} \tag{5}$$

A similar thing occurs when setting $x=0$ and setting $y=0$.

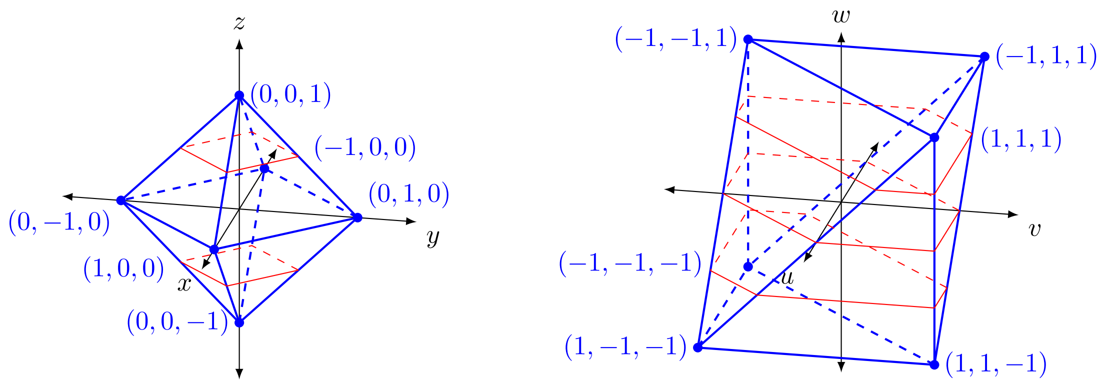

Figure 2: The domain of integration in (x,y,z)-coordinates and in (u,v,w)-coordinates.

The boundary faces are the planes

$$\begin{cases} x+y+z=1, \\ x+y+z=-1, \\ x+y-z=1, \\ x+y-z=-1, \\ x-y+z=1, \\ x-y+z=-1, \\ x-y-z=1, \\ x-y-z=-1. \end{cases} \tag{6}$$

Consider the change of variables

$$\begin{cases} u=x+y+z, \\ v=x+y-z, \\ w=x-y-z \end{cases} \longleftrightarrow \begin{cases} x=\frac{1}{2}(u+w), \\ y=\frac{1}{2}(v-w), \\ z=\frac{1}{2}(u-v). \end{cases} \tag{7}$$

The domain then becomes

$$\begin{cases} -1 \leqslant u \leqslant 1, \\ -1 \leqslant v \leqslant 1, \\ -1 \leqslant w \leqslant 1, \\ -1 \leqslant u - v + w \leqslant 1. \end{cases} \tag{8}$$

and the Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} = -\frac{1}{4}. \tag{9}$$

Hence

$$\begin{aligned} \iiint_D (x + y + z)^4 \, dV &= \iiint_{\substack{-1 \leqslant u,v,w \leqslant 1 \\ -1 \leqslant u-v+w \leqslant 1}} \frac{u^4}{4} \, dw \, dv \, du \\ &= \underbrace{\iiint_{-1 \leqslant u,v,w \leqslant 1} \frac{u^4}{4} \, dw \, dv \, du}_{(\mathbf{A})} - \underbrace{\iiint_{\substack{-1 \leqslant u,v,w \leqslant 1 \\ u-v+w \leqslant -1}} \frac{u^4}{4} \, dw \, dv \, du}_{(\mathbf{B})} - \underbrace{\iiint_{\substack{-1 \leqslant u,v,w \leqslant 1 \\ u-v+w \geqslant 1}} \frac{u^4}{4} \, dw \, dv \, du.}_{(\mathbf{C})} \end{aligned} \tag{10}$$

(We do this because its easier to describe the regions missing from the cube.)

First, we can check that

$$(\mathbf{A}) = \iiint_{-1 \leqslant u,v,w \leqslant 1} \frac{u^4}{4} \, dw \, dv \, du = \ldots = \frac{2}{5}. \tag{11}$$

For $(\mathbf{B})$ and $(\mathbf{C})$, we notice that we have to integrate over the leftover regions of the cube.

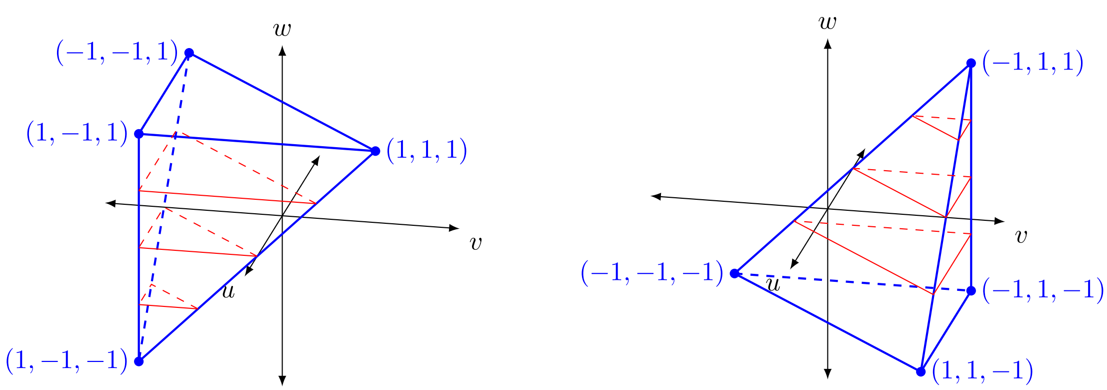

Figure 3: The domains of integration for the terms $(\mathbf{B})$ and $(\mathbf{C})$.

We can check that the angled boundary face for $(\mathbf{B})$ is given by $u - v + w = 1$. Likewise, the angled boundary face for $(\mathbf{C})$ is $u - v + w = -1$. These regions can be described as one of our special domains. In particular, for $(\mathbf{B})$, the region is given by

$$\{(u, v, w) \in \mathbb{R}^3 \mid -1 \leqslant u \leqslant 1, -u \leqslant w \leqslant 1, -1 \leqslant v \leqslant u + w - 1\}. \tag{12}$$

Similarly, the region for $(\mathbf{C})$ is given by

$$\{(u, v, w) \in \mathbb{R}^3 \mid -1 \leqslant u \leqslant 1, 1 \leqslant w \leqslant -u, u + w - 1 \leqslant v \leqslant 1\}. \tag{13}$$

The integrals are then

$$(\mathbf{B}) = \int_{-1}^{1}\int_{-u}^{1}\int_{-1}^{u+w-1} \frac{u^4}{4}\, dv\, dw\, du = \ldots = \frac{3}{35}. \tag{14}$$

By the symmetry $(u, v, w) \longleftrightarrow (-u, -v, -w)$ we can see that

$$(\mathbf{C}) = (\mathbf{B}) = \frac{3}{35}. \tag{15}$$

Thus putting everything together, we get

$$\iiint_D (x + y + z)^4\, dV = (\mathbf{A}) - (\mathbf{B}) - (\mathbf{C}) = \frac{2}{5} - \frac{3}{35} - \frac{3}{35} = \frac{8}{35}. \tag{16}$$

□

1 Vector Analysis

We now move on to the second part of the course.

Remark 3. We will use an arrow $\vec{\cdot}$ to denote general vectors and a hat $\hat{\cdot}$ for unit vectors.

The vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the unit vectors in the $x-$, $y-$, and z-directions respectively.

The length of a vector $\vec{v}$ will be denoted $\|\vec{v}\|$ to avoid confusion with the absolute value operator $|\cdot|$.

1.1 Path (Line) Integrals in $\mathbb{R}^2$ and $\mathbb{R}^3$

Definition 4 (Path Integral)**.** The *integral of a function f along a curve (or path, or line) C* with parameterization

$$\begin{aligned} \vec{r}\colon [a, b] &\to \mathbb{R}^3 \\ t &\mapsto \big(x(t), y(t), z(t)\big) \end{aligned}$$

is defined as

$$\int_C f(\vec{r})\, ds = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f\big(\vec{r}(t_k)\big) \cdot \Delta s_k, \tag{17}$$

where

- $P\colon a = c_0 < c_1 < \ldots < c_n = b$ is a partition of $[a, b]$,
- $t_k \in [c_{k-1}, c_k]$,
- $\Delta s_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2 + (\Delta z_k)^2}$.

Here ds represents the *length element* of the curve C.

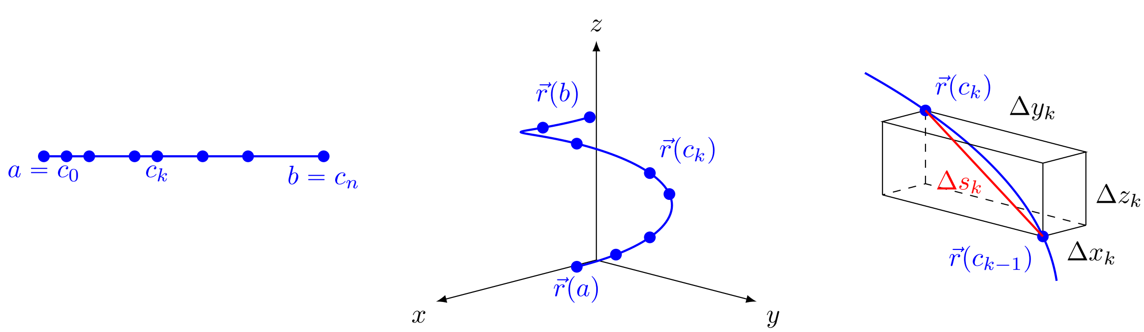

Figure 4: A partition P of the interval $[a, b]$ segments the curve C.

Remark 5. If $f \equiv 1$, then the resulting line integral

$$\int_C 1\, ds \tag{18}$$

is the *arc-length* of C.

Remark 6. The path integral is well-defined, that is, the definition is independent on the choice of partitions P and parameterization $\vec{r}$.

Of course, we would like to not have to use partitions and Riemann sums to practically compute these path integrals. The following formula will help with this.

Proposition 7. *Let f be a function along a curve C with parameterization $\vec{r}(t) = \big(x(t), y(t), z(t)\big)$, then*

$$\int_C f(\vec{r})\, ds = \int_a^b f\big(\vec{r}(t)\big) \cdot \|\vec{r'}(t)\|\, dt \tag{19}$$

where $\vec{r'}(t) = \big(x'(t), y'(t), z'(t)\big)$.

Loosely speaking, we can write

$$\begin{aligned}\Delta s_k &= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2 + (\Delta z_k)^2} \\ &= \sqrt{\left(\frac{\Delta x_k}{\Delta t_k}\right)^2 + \left(\frac{\Delta y_k}{\Delta t_k}\right)^2 + \left(\frac{\Delta z_k}{\Delta t_k}\right)^2} \cdot \Delta t_k \\ &\cong \sqrt{x'(t_k)^2 + y'(t_k)^2 + z'(t_k)^2} \cdot \Delta t_k \\ &= \|\vec{r'}(t_k)\| \cdot \Delta t_k. \end{aligned} \tag{20}$$

Thus this formula conceptually makes sense when compared with the Riemann sum definition of the path integral.

Remark 8 (Arc-Length Element)**.** The quantity

$$ds = \|\vec{r'}(t)\|\, dt \tag{21}$$

is usually referred to as the *arc-length element* where

$$\vec{r'}(t) = \big(x'(t), y'(t), z'(t)\big) \text{ and } \|\vec{r'}(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}. \tag{22}$$

Remark 9 (Change of Parameterization)**.** Suppose we parameterize the curve C by another parameter $\widetilde{t}$:

$$t \in [a, b] \longleftrightarrow \widetilde{t} \in [\widetilde{a}, \widetilde{b}] \tag{23}$$

and suppose that the change of parameter is *orientation-preserving* so

$$\frac{\partial t}{\partial \widetilde{t}} > 0 \text{ and } \frac{\partial \widetilde{t}}{\partial t} > 0. \tag{24}$$

We see that the *arc-length element* ds then satisfies

$$\|\vec{r'}(t)\|\, dt = \left\|\frac{d\vec{r}}{dt}\right\| dt = \left\|\frac{d\vec{r}}{d\widetilde{t}} \cdot \frac{d\widetilde{t}}{dt}\right\| dt = \left\|\frac{d\vec{r}}{d\widetilde{t}}\right\| \cdot \left|\frac{d\widetilde{t}}{dt}\right| dt = \left\|\frac{d\vec{r}}{d\widetilde{t}}\right\| d\widetilde{t} = \|\vec{r'}(\widetilde{t})\|\, d\widetilde{t}. \tag{25}$$

From this, we see that ds and the path integral $\int_C f(\vec{r})\, ds$ is independent on our choice of parameterization (as long as we keep the orientation).

If the change of parameter is *orientation-reversing*, so

$$\frac{\partial t}{\partial \widetilde{t}} < 0 \text{ and } \frac{\partial \widetilde{t}}{\partial t} < 0, \tag{26}$$

we instead introduce a negative sign

$$\|\vec{r'}(t)\|\, dt = \left\|\frac{d\vec{r}}{dt}\right\| dt = \left\|\frac{d\vec{r}}{d\widetilde{t}} \cdot \frac{d\widetilde{t}}{dt}\right\| dt = \left\|\frac{d\vec{r}}{d\widetilde{t}}\right\| \cdot \left|\frac{d\widetilde{t}}{dt}\right| dt = -\left\|\frac{d\vec{r}}{d\widetilde{t}}\right\| d\widetilde{t} = -\|\vec{r'}(\widetilde{t})\|\, d\widetilde{t}. \tag{27}$$

Since we have flipped the orientation, the bounds on integration will also flip, giving another factor of -1. These will cancel and again show that the path integral $\int_C f(\vec{r})\, ds$ is independent (even if we change orientations).

Remark 10 (Piecewise Differentiable Curves)**.** If the parameterization $\vec{r}$ of the curve C is only piecewise differentiable, then we can compute the path integral over C as a sum of path integrals over the differentiable sub-curves C_k.

Phrased differently, if we write

$$[a,b] = [c_0, c_1] \cup \ldots \cup [c_{k-1}, c_k] \cup \ldots [c_{n-1}, c_n], \tag{28}$$

such that $\vec{r}\big|_{[c_{k-1},c_k]}$ is differentiable, then

$$\int_C f(\vec{r})\, ds = \sum_{k=1}^{n} \int_{c_{k-1}}^{c_k} f\big(\vec{r}(t)\big) \cdot \|\vec{r}'(t)\|\, dt. \tag{29}$$

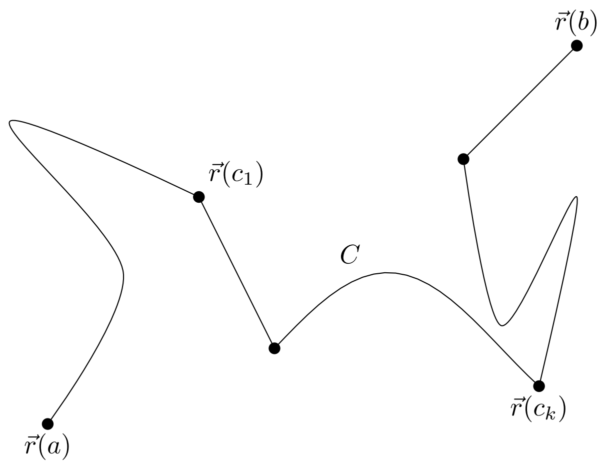

Figure 5: A piecewise differentiable curve C.

Example 11. Let $f(x,y,z) = x - 3y^2 + z$ and C be the (straight) line segment joining the origin at $(1,1,1)$. Find

$$\int_C f(x,y,z)\, ds. \tag{30}$$

Solution. Parameterize the curve C by

$$\begin{aligned} \vec{r}(t) &= (0,0,0) + t\big[(1,1,1) - (0,0,0)\big] \\ &= (t,t,t), \quad t \in [0,1]. \end{aligned} \tag{31}$$

With this choice, we have $\vec{r}'(t) = (1,1,1)$ for $t \in [0,1]$ and so the integral becomes

$$\begin{aligned} \int_C f(\vec{r})\, ds &= \int_0^1 f(t,t,t) \cdot \|\vec{r}'(t)\|\, dt \\ &= \int_0^1 \Big(t - 3t^2 + t\Big) \cdot \sqrt{3}\, dt \\ &= \ldots = 0. \qquad [\textbf{Exercise} : \text{Check this}] \end{aligned} \tag{32}$$

□

Example 12. Let C be a plane curve in $\mathbb{R}^2$ (*i.e.* $z(t) = 0$) and suppose it has two parameterizations:

$$\vec{r_1}(t) = (\cos t, \sin t), \quad t \in [-\frac{\pi}{2}, \frac{\pi}{2}], \tag{33}$$

$$\vec{r_2}(t) = (\sqrt{1-t^2}, -t), \quad t \in [-1, 1]. \tag{34}$$

Let $f(x,y) = x$. Find $\int_C f(x,y)\, ds$.

Solution. Here we omit the z-variable since C is a plane curve.

Using the first parameterization, we have

$$\vec{r_1'}(t) = (-\sin t, \cos t) \implies \|\vec{r_1'}(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1, \tag{35}$$

and so

$$\begin{aligned}\int_C f\,ds &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot 1\,dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\,dt = 2.\end{aligned} \tag{36}$$

Using the second parameterization, we have

$$\vec{r_2'}(t) = \left(-\frac{t}{\sqrt{1-t^2}}, -1\right) \implies \|\vec{r_2'}(t)\| = \sqrt{\frac{t^2}{1-t^2}+1} = \frac{1}{\sqrt{1-t^2}}. \tag{37}$$

The formula then gives

$$\begin{aligned}\int_C f\,ds &= \int_{-1}^{1} \sqrt{1-t^2} \cdot \frac{1}{\sqrt{1-t^2}}\,dt \\ &= \int_{-1}^{1} 1\,dt = 2.\end{aligned} \tag{38}$$

This agrees with the fact that the path integral is independent of the choice of parameterization. Note here that the two parameterizations here run in different directions.

Note also that we can interpret this integral as a (signed) area. □

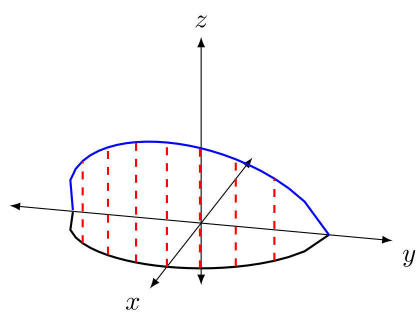

Figure 6: A scalar path integral over a plane curve can also be interpreted as a (signed) area.

Proposition 13. *If C is a piecewise smooth curve made by joining $C_1, C_2, \ldots, C_n$* end-to-end, *then*

$$\int_C f\,ds = \sum_{k=1}^{n} \int_{C_k} f\,ds. \tag{39}$$

Proof. Omitted. □

Remark 14. *End-to-end* in the previous proposition means that the terminal point of C_{k-1} is the initial point of C_k.

Example 15. Let $f(x,y,z) = x - 3y^2 + z$ again. Let C_1, C_2, and C_3 be the (straight) line segments as in the figure

In a previous example, we showed that

$$\int_{C_1} f\,ds = 0. \tag{40}$$

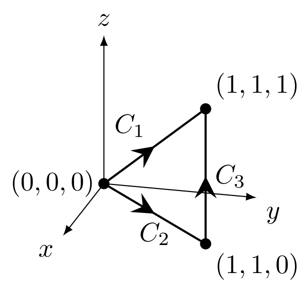

Figure 7: Two different paths from the origin $(0, 0, 0)$ to the point $(1, 1, 1)$.

One can similarly calculate that

$$\int_{C_2 \cup C_3} f\, ds = \int_{C_2} f\, ds + \int_{C_3} f\, ds = -\frac{\sqrt{2}}{2} - \frac{3}{2}. \qquad \text{[Exercise : Check this]} \tag{41}$$

From this, we can observe that

$$\int_{C_1} f\, ds \neq \int_{C_2 \cup C_3} f\, ds \tag{42}$$

even though the curves C_1 and $C_2 \cup C_3$ have the *same start and end points*. This is different from the case in 1 dimension.

From the previous example, we have the conclusion that *a path integral depends not only just on the start and end points, but also the path taken.*

1.2 Vector Fields

Definition 16. Let $D \subseteq \mathbb{R}^2$ (or $\mathbb{R}^3$) be a domain. A *vector field* on D is a map $\vec{F} \colon D \to \mathbb{R}^2$ (or respectively $\mathbb{R}^3$).

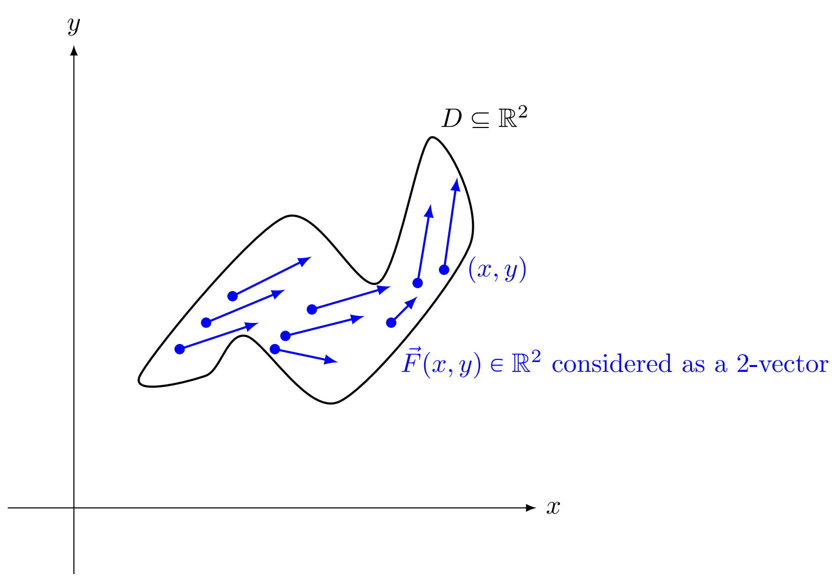

Figure 8: A vector field $\vec{F}$ on a domain $D \subseteq \mathbb{R}^2$.

We can write

$$\vec{F}(x, y) = M(x, y)\,\hat{\mathbf{i}} + N(x, y)\,\hat{\mathbf{j}} \tag{43}$$

for the $\mathbb{R}^2$ case and

$$\vec{F}(x,y,z) = M(x,y,z)\,\hat{\mathbf{i}} + N(x,y,z)\,\hat{\mathbf{j}} + L(x,y,z)\,\hat{\mathbf{k}} \tag{44}$$

for the $\mathbb{R}^3$ case. Here M, N, and L are called the *components* of $\vec{F}$.

Example 17. Consider $\vec{F}(x,y) = \frac{-y\,\hat{\mathbf{i}}+x\,\hat{\mathbf{j}}}{\sqrt{x^2+y^2}}$ on $\mathbb{R}^2\backslash\{(0,0)\}$. Using polar coordinates, we can write this as

$$\vec{F}(r,\theta) = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}. \tag{45}$$

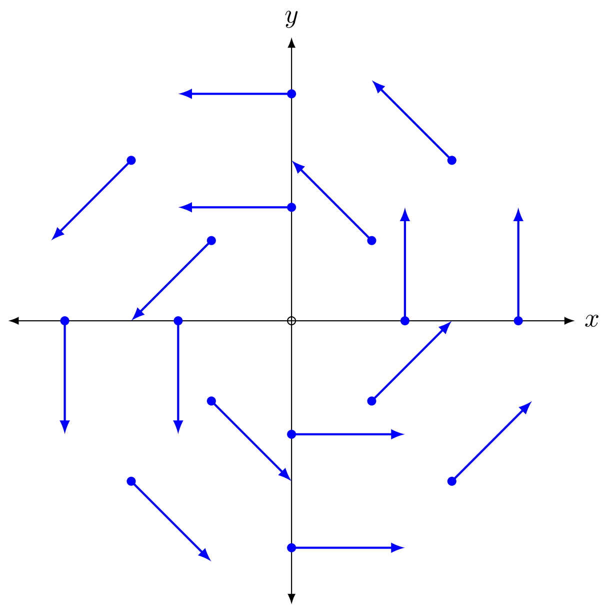

Figure 9: The vector field $\vec{F}(r,\theta) = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$.

We can check that $\vec{F}$ has the properties that

$$\|\vec{F}(x,y)\| = 1 \tag{46}$$

and that

$$\vec{F}(x,y) \perp \vec{r}(x,y), \tag{47}$$

where $\vec{r}(x,y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} = r(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}})$ is the position vector field.

Definition 18 (Gradient Vector Field of a Function)**.** Given a function $f(x,y)$ on $\mathbb{R}^2$, we define its *gradient vector field* $\vec{\nabla}f(x,y)$ to be the vector field

$$\vec{\nabla}f(x,y) = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}}. \tag{48}$$

For a function $f(x,y,z)$ on $\mathbb{R}^3$, we similarly define its *gradient vector field* $\vec{\nabla}f(x,y,z)$ to be the vector field

$$\vec{\nabla}f(x,y,z) = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}}. \tag{49}$$

Example 19. Let $f(x,y) = \frac{1}{2}(x^2+y^2)$. Then

$$\vec{\nabla}f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (x,y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} = \vec{r}(x,y). \tag{50}$$

Let $f(x,y,z) = x$. Then

$$\vec{\nabla}f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (1,0,0) = \hat{\mathbf{i}}. \tag{51}$$

Remark 20. The radial vector field $\vec{r}(x, y)$ and also the component vector fields $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are gradient vector fields. This notion will important for the future.

(End of Lecture 9 – Oct 6)