MATH2020A Lecture 8 Notes

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Last time, we looked at arbitrary changes of variables $(x, y) = \Phi(u, v)$. Part of this involved finding out how our transformation affects areas, since we defined integration using Riemann sums. We saw that this area scaling was related to the Jacobian determinant $\frac{\partial(x,y)}{\partial(u,v)}$ of the change of coordinates.

Example 1. Compute

$$I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}} \, dx \, dy \tag{1}$$

Solution. Consider the change of variables

$$\begin{cases} u = \sqrt{xy}, \\ v = \sqrt{\frac{y}{x}} \end{cases} \longleftrightarrow \begin{cases} x = \frac{u}{v}, \\ y = uv. \end{cases} \tag{2}$$

We see that the bounds of integration are

$$\begin{cases} x = \frac{1}{y}, \\ x = y, \\ y = 1, \\ y = 2 \end{cases} \longleftrightarrow \begin{cases} u = 1, \\ v = 1, \\ v = \frac{1}{u}, \\ v = \frac{2}{u}. \end{cases} \tag{3}$$

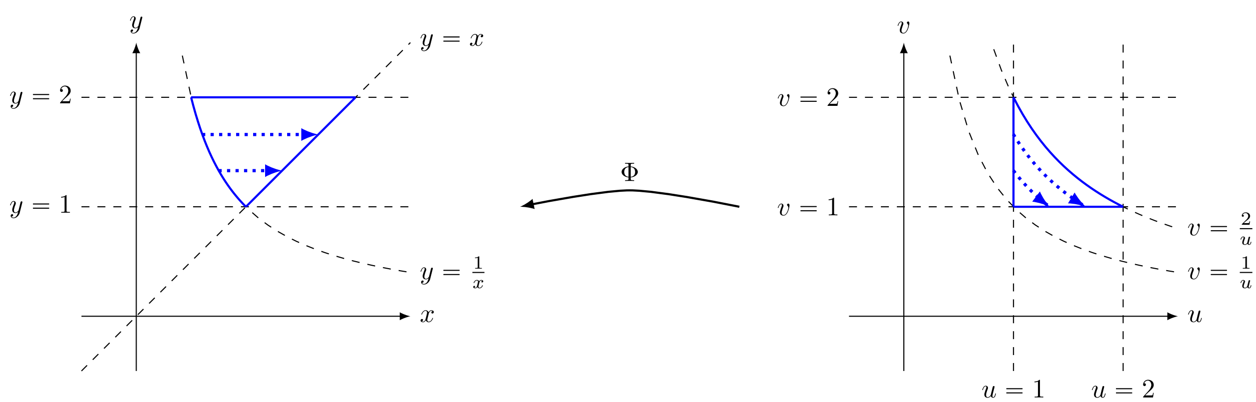

Figure 1: A change of coordinates.

Our Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{v} & -\frac{1}{v^2} \\ v & u \end{bmatrix} = \frac{2u}{v}. \tag{4}$$

Plugging everything in, we get

$$\begin{aligned}
I &= \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}}\, dx\, dy \\
&= \int_1^2 \int_1^{\frac{2}{u}} ve^u \cdot \left|\frac{2u}{v}\right| dv\, du \\
&= \int_1^2 \int_1^{\frac{2}{u}} 2ue^u\, dv\, du \\
&= \int_1^2 2ue^u \left(\int_1^{\frac{2}{u}} dv \right) du \\
&= \int_1^2 2ue^u \left(\frac{2}{u} - 1 \right) du \\
&= \ldots = 2e(e-2). \qquad \text{[Exercise : Check this]} \qquad (5)
\end{aligned}$$

□

Example 2 (Volume of an Ellipsoid Revisited)**.** Recall that in a previous example, we wanted to compute the volume of the ellipsoid

$$D = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1 \right\}, \quad (a, b, c > 0). \tag{6}$$

Last time, we set up the integral as

$$\text{Volume}(D) = 8 \cdot \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx. \tag{7}$$

To make this computation easier, we can use the change of variables

$$\begin{cases} u = \frac{x}{a}, \\ v = \frac{y}{b}, \\ w = \frac{z}{c} \end{cases} \longleftrightarrow \begin{cases} x = au, \\ y = bv, \\ z = cw. \end{cases} \tag{8}$$

The new domain becomes

$$R = \{ (u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 \leqslant 1 \} \tag{9}$$

which is the unit ball in (u, v, w)-space.

The Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = abc. \tag{10}$$

After the change of coordinates, the integral becomes

$$\begin{aligned}
\text{Volume}(D) &= 8 \cdot \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx && (11)\\
&= 8 \cdot \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dw\, dv\, du && (12)\\
&= 8abc \cdot \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} dw\, dv\, du && (13)\\
&= abc \cdot (\text{Volume of unit ball in } (u,v,w)\text{-coordinates}) && (14)\\
&= \frac{4\pi}{3} abc. && (15)
\end{aligned}$$

Example 3. Let

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid |x| + |y| + |z| \leqslant 1\}. \tag{16}$$

Evaluate

$$\iiint_D (x+y+z)^4 \, dV. \tag{17}$$

Remark 4. Before we begin, we first note that we can use symmetries to make things slightly easier. We have to be careful since we have to make sure that the integrand does not change under the symmetries we use.

For example, consider the symmetry $(x,y,z) \longleftrightarrow (-x,-y,-z)$. We can check that

$$\big((-x)+(-y)+(-z)\big)^4 = \big(-(x+y+z)\big)^4 = (x+y+z)^4 \tag{18}$$

and so the integrand respects this symmetry.

On the other hand, consider $(x,y,z) \longleftrightarrow (x,y,-z)$. We compute that

$$\big(x+y+(-z)\big)^4 \neq (x+y+z)^4 \tag{19}$$

in general and so we cannot reduce using this symmetry.

What this means here is that we cannot use the first octant trick from our earlier examples since we need the integrand to satisfy all eight symmetries given by reflecting over coordinate planes.

Solution. Let us first try to get an idea of the shape we are integrating over. Setting $z = 0$, we have $|x| + |y| \leqslant 1$ which looks like a (rotated) square.

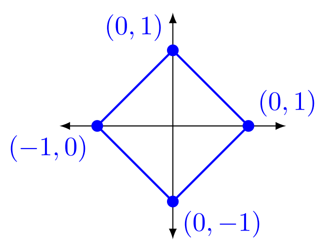

Figure 2: Projecting the domain D onto the xy-plane.

The boundary lines here are

$$\begin{cases} x + y = 1, \\ x + y = -1, \\ x - y = 1, \\ x - y = -1. \end{cases} \tag{20}$$

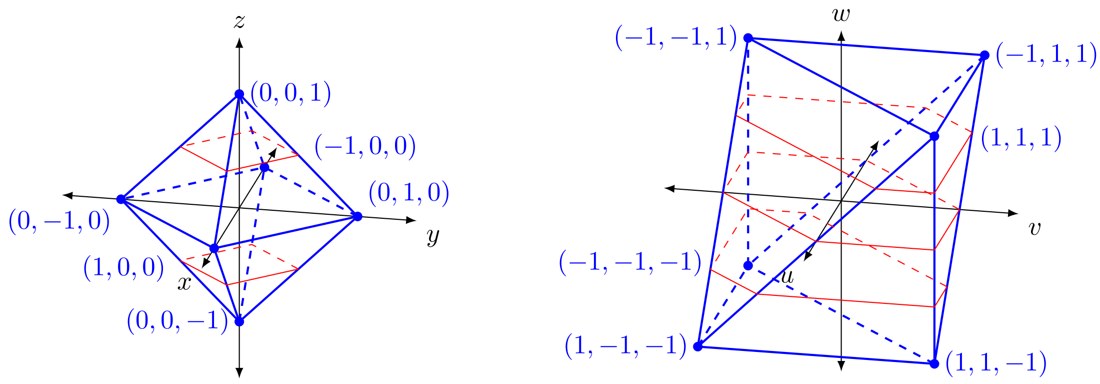

Figure 3: The domain of integration in (x, y, z)-coordinates and in (u, v, w)-coordinates.

A similar thing occurs when setting $x = 0$ and setting $y = 0$.

The boundary faces are the planes

$$\begin{cases} x + y + z = 1, \\ x + y + z = -1, \\ x + y - z = 1, \\ x + y - z = -1, \\ x - y + z = 1, \\ x - y + z = -1, \\ x - y - z = 1, \\ x - y - z = -1. \end{cases} \tag{21}$$

Consider the change of variables

$$\begin{cases} u = x + y + z, \\ v = x + y - z, \\ w = x - y - z \end{cases} \longleftrightarrow \begin{cases} x = \frac{1}{2}(u + w), \\ y = \frac{1}{2}(v - w), \\ z = \frac{1}{2}(u - v). \end{cases} \tag{22}$$

The domain then becomes

$$\begin{cases} -1 \leqslant u \leqslant 1, \\ -1 \leqslant v \leqslant 1, \\ -1 \leqslant w \leqslant 1, \\ -1 \leqslant u - v + w \leqslant 1. \end{cases} \tag{23}$$

and the Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} = -\frac{1}{4}. \tag{24}$$

Hence

$$\iiint_D (x+y+z)^4 \, dV = \iiint_{\substack{-1\leqslant u,v,w\leqslant 1 \\ -1\leqslant u-v+w\leqslant 1}} \frac{u^4}{4}\, dw\, dv\, du$$

$$= \underbrace{\iiint_{-1\leqslant u,v,w\leqslant 1} \frac{u^4}{4}\, dw\, dv\, du}_{(\mathbf{A})} - \underbrace{\iiint_{\substack{-1\leqslant u,v,w\leqslant 1 \\ u-v+w\leqslant -1}} \frac{u^4}{4}\, dw\, dv\, du}_{(\mathbf{B})} - \underbrace{\iiint_{\substack{-1\leqslant u,v,w\leqslant 1 \\ u-v+w\geqslant 1}} \frac{u^4}{4}\, dw\, dv\, du}_{(\mathbf{C})}. \tag{25}$$

(We do this because its easier to describe the regions missing from the cube.)

First, we can check that

$$(\mathbf{A}) = \iiint_{-1\leqslant u,v,w\leqslant 1} \frac{u^4}{4}\, dw\, dv\, du = \ldots = \frac{2}{5}. \tag{26}$$

TO BE CONTINUED □

---**(End of Lecture 8 – Oct 2)**