MATH2020A Lecture 7 Notes

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Last time, we looked at two convenient coordinate systems – cylindrical coordinates and spherical coordinates. We have the following conversions betwen them:

$$\begin{array}{ccccc} x & = & r\cos\theta & = & \rho\sin\varphi\cos\theta \\ y & = & r\sin\theta & = & \rho\sin\varphi\sin\theta \\ z & = & z & = & \rho\cos\varphi \\ dx\,dy\,dz & = & r\,dr\,d\theta\,dz & = & \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\ \text{Cartesian} & \longleftrightarrow & \text{Cylindrical} & \longleftrightarrow & \text{Spherical} \end{array} \tag{1}$$

Example 1 (Volume of an Ice Cream Cone)**.** Find the volume of the ice cream cone I as shown in the figure.

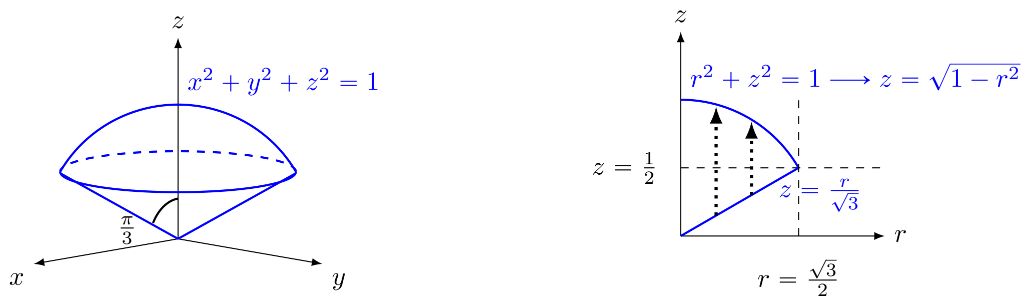

Figure 1: An ice cream cone and its cross-section.

Solution. We can recognize that this is a solid obtained by rotating around the z-axis from $\theta = 0$ to $\theta = 2\pi$, and so by keeping θ fixed, we obtain a uniform slice of the solid.

By Fubini's Theorem, the volume of the ice cream cone is given in cylindrical coordinates by the integral

$$\begin{aligned} \text{Volume}(I) = \iiint_I dV &= \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta \\ &= 2\pi\cdot\int_0^{\frac{\sqrt{3}}{2}}\left(\sqrt{1-r^2}-\frac{r}{\sqrt{3}}\right)\cdot r\,dz \\ &= 2\pi\cdot\left[-\frac{1}{3}\sqrt{1-r^2}^{\frac{3}{2}}-\frac{r^3}{3\sqrt{3}}\Big|_{r=0}^{r=\frac{\sqrt{3}}{2}}\right] \\ &= 2\pi\cdot\left(-\frac{1}{3}\cdot\frac{1}{8}-\frac{1}{8}+\frac{1}{3}\right) = \frac{\pi}{3}. \end{aligned} \tag{2}$$

Alternatively, we can use spherical coordinates by recognizing I as the intersection of the unit ball $x^2+y^2+z^2 \leqslant 1$ and the cone $\varphi \leqslant \frac{\pi}{3}$ (remember that φ is being measured from the positive z-axis).

The volume is then given by

$$
\begin{aligned}
\text{Volume}(I) &= \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 1\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\
&= \left[\int_0^{2\pi} d\theta\right]\cdot\left[\int_0^{\frac{\pi}{3}}\sin\varphi\,d\varphi\right]\cdot\left[\int_0^1\rho^2\,d\rho\right] \\
&= (2\pi)\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{1}{3}\right) = \frac{\pi}{3}. \qquad (3)
\end{aligned}
$$

Of course, this aligns with our previous calculation. □

Example 2. Let $f(x, y, z)$ be the function defined by

$$
f(x,y,z) = \begin{cases} \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}, & \text{if } (x,y,z) \neq 0, \\ 0, & \text{if } (x,y,z) = (0,0,0). \end{cases} \qquad (4)
$$

(This function is continuous everywhere except maybe at the origin $(0, 0, 0)$.) Also let D be the intersection of the unit ball centered at the origin with the first octant. Find the *average* of the function f over D.

Solution. The region D can be represented in spherical coordinates as

$$
D = \left\{(\rho,\varphi,\theta)\in\mathbb{R}^3 \mid 0\leqslant\rho\leqslant 1, 0\leqslant\varphi\leqslant\frac{\pi}{2}, 0\leqslant\theta\leqslant\frac{\pi}{2}\right\}. \qquad (5)
$$

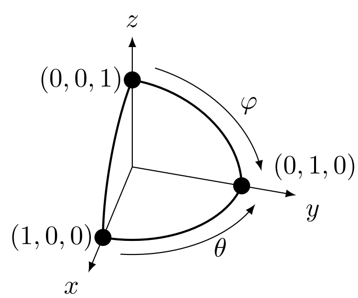

Figure 2: The intersection of the unit ball and the first octant.

We can also write the function f (when $(x, y, z) \neq (0, 0, 0)$) in terms of ρ, φ, and θ as

$$
f(x,y,z) = \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} = \frac{\rho^2\sin^2\varphi}{\rho} = \rho\sin^2\varphi. \qquad (6)
$$

(From this, we can see that $f \to 0$ as $\rho \to 0$, which shows that f is indeed continuous, even at the origin.)

We note that the volume of the region D is

$$
\text{Volume}(D) = \frac{1}{8}\cdot\text{Volume of Unit Ball} = \frac{1}{8}\cdot\frac{4\pi}{3} = \frac{\pi}{6}. \qquad (7)
$$

Our desired integral then becomes

$$\begin{aligned}
\text{Average Value of } f \text{ on } D &= \frac{1}{\text{Volume}(D)} \cdot \iiint_D f(x,y,z)\,dV \\
&= \frac{6}{\pi} \cdot \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho \sin^2\varphi \cdot \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta \\
&= \frac{6}{\pi} \cdot \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho^3 \sin^3\varphi \, d\rho\, d\varphi\, d\theta \\
&= \left(\frac{6}{\pi}\right) \cdot \left(\frac{\pi}{2}\right) \cdot \left[\int_0^{\frac{\pi}{2}} \sin^3\varphi\, d\varphi\right] \cdot \left[\int_0^1 \rho^3\, d\rho\right] \\
&= \ldots = \frac{1}{2}. \qquad [\textbf{Exercise} : \text{Check this}] \qquad (8)
\end{aligned}$$

□

Example 3 (Improper Integrals)**.** Let

$$f(x,y,z) = \frac{1}{x^2+y^2+z^2} = \frac{1}{\rho^2} \tag{9}$$

and

$$g(x,y,z) = \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} = \frac{1}{\rho^3}. \tag{10}$$

Both f and g are unbounded as $\rho \to 0$. Consider both functions over the unit ball $B = \{(\rho,\varphi,\theta) \in \mathbb{R}^3 \mid 0 \leqslant \rho \leqslant 1\}$.

1. Does

$$\lim_{\epsilon\to 0} \iiint_{B\setminus B_\epsilon} f(x,y,z)\,dV \tag{11}$$

   exist? (Here $B_\epsilon = \{(\rho,\varphi,\theta) \in \mathbb{R}^3 \mid 0 \leqslant \rho \leqslant \epsilon\}$ is the ball of radius ϵ.)

2. Does

$$\lim_{\epsilon\to 0} \iiint_{B\setminus B_\epsilon} g(x,y,z)\,dV \tag{12}$$

   exist?

Solution. 1. We compute directly that

$$\begin{aligned}
\lim_{\epsilon\to 0} \iiint_{B\setminus B_\epsilon} f(x,y,z)\,dV &= \lim_{\epsilon\to 0} \int_0^{2\pi} \int_0^{\pi} \int_\epsilon^1 \frac{1}{\rho^2} \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \\
&= \lim_{\epsilon\to 0} \left[\int_0^{2\pi} d\theta\right] \cdot \left[\int_0^{\pi} \sin\varphi\, d\varphi\right] \cdot \left[\int_\epsilon^1 d\rho\right] \\
&= \lim_{\epsilon\to 0} (2\pi)\cdot(2)\cdot(1-\epsilon) = 4\pi. \qquad (13)
\end{aligned}$$

2. On the other hand, we see that

$$\begin{aligned}
\lim_{\epsilon\to 0} \iiint_{B\setminus B_\epsilon} g(x,y,z)\,dV &= \lim_{\epsilon\to 0} \int_0^{2\pi} \int_0^{\pi} \int_\epsilon^1 \frac{1}{\rho^3} \cdot \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta \\
&= \lim_{\epsilon\to 0} \left[\int_0^{2\pi} d\theta\right] \cdot \left[\int_0^{\pi} \sin\varphi\, d\varphi\right] \cdot \left[\int_\epsilon^1 \frac{1}{\rho}\, d\rho\right] \\
&= \lim_{\epsilon\to 0} (2\pi)\cdot(2)\cdot\left(\ln\frac{1}{\epsilon}\right) = \text{DNE}.
\end{aligned}$$

□

Remark 4. Using the example above, we say that $f = \frac{1}{\rho^2}$ is *integrable* over B (in the sense of "improper integrals"). The function $g = \frac{1}{\rho^3}$ is said to be *non-integrable* over B.

Exercise :

1. Determine all $\beta > 0$ such that $f = \frac{1}{\rho^\beta}$ is *integrable* over $B \subseteq \mathbb{R}^3$.
2. Determine all $\beta > 0$ such that $f = \frac{1}{r^\beta}$ is *integrable* over $\{r \leqslant 1\} \subseteq \mathbb{R}^2$.
3. Determine all $\beta > 0$ such that $f = \frac{1}{|x|^\beta}$ is *integrable* over $\{x \leqslant 1\} \subseteq \mathbb{R}^1$.

0.1 Applications of Multiple Integrals

In terms of applications, we often use the following:

Definition 5 (First Moments, Mass, and Centroids)**.** Let R be a region in $\mathbb{R}^2$ with *density* function $\delta(x, y)$ we define:

- the *first moment about the y-axis* of R as
$$M_y = \iint\limits_R x \cdot \delta(x, y)\, dA; \tag{14}$$
- the *first moment about the x-axis* of R as
$$M_x = \iint\limits_R y \cdot \delta(x, y)\, dA; \tag{15}$$
- the *mass* of R as
$$M = \iint\limits_R \delta(x, y)\, dA; \tag{16}$$
- the *center of mass (or centroid)* of R as
$$(\overline{x}, \overline{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right); \tag{17}$$

We also have 3-dimensional analogues of these

Definition 6 (First Moments, Mass, and Centroids II)**.** Let R be a region in $\mathbb{R}^3$ with *density* function $\delta(x, y, z)$ we define:

- the *first moment about the yz-plane* of R as
$$M_{yz} = \iiint\limits_R x \cdot \delta(x, y, z)\, dV; \tag{18}$$
- the *first moment about the xz-plane* of R as
$$M_{xz} = \iiint\limits_R y \cdot \delta(x, y, z)\, dV; \tag{19}$$
- the *first moment about the xy-plane* of R as
$$M_{xy} = \iiint\limits_R z \cdot \delta(x, y, z)\, dV; \tag{20}$$

- the *mass* of R as

$$M = \iiint\limits_R \delta(x, y, z)\, dV; \tag{21}$$

- the *center of mass (or centroid)* of R as

$$(\overline{x}, \overline{y}, \overline{z}) = \Big(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\Big). \tag{22}$$

We also can define *moments of inertia* in both cases:

Definition 7 (Moments of Inertia)**.** Let R be a region in $\mathbb{R}^2$ with *density* function $\delta(x, y)$ we define:

- the *moment of inertia about the y-axis* of R as

$$I_y = \iint\limits_R x^2 \cdot \delta(x, y)\, dA; \tag{23}$$

- the *moment of inertia about the x-axis* of R as

$$I_x = \iint\limits_R y^2 \cdot \delta(x, y)\, dA; \tag{24}$$

- the *moment of inertia about the line L* of R as

$$I_L = \iint\limits_R r(x, y)^2 \cdot \delta(x, y)\, dA, \tag{25}$$

  where $r(x, y)$ is the distance between the point (x, y) and the line L;

- the *moment of inertia about the origin* of R as

$$I_O = \iint\limits_R (x^2 + y^2) \cdot \delta(x, y)\, dA. \tag{26}$$

Definition 8 (Moments of Inertia II)**.** Let R be a region in $\mathbb{R}^3$ with *density* function $\delta(x, y, z)$ we define:

- the *moment of inertia about the z-axis* of R as

$$I_z = \iiint\limits_R (x^2 + y^2) \cdot \delta(x, y, z)\, dV; \tag{27}$$

- the *moment of inertia about the y-axis* of R as

$$I_y = \iiint\limits_R (x^2 + z^2) \cdot \delta(x, y, z)\, dV; \tag{28}$$

- the *moment of inertia about the x-axis* of R as

$$I_x = \iiint\limits_R (y^2 + z^2) \cdot \delta(x, y, z)\, dV; \tag{29}$$

- the *moment of inertia about the line L* of R as

$$I_L = \iiint\limits_R r(x, y, z)^2 \cdot \delta(x, y, z)\, dV, \tag{30}$$

  where $r(x, y, z)$ is the distance between the point (x, y, z) and the line L.

Example 9. Consider the region $D = \{(x, y, z) \in \mathbb{R}^3 \mid r^2 \leqslant x^2 + y^2 + z^2 \leqslant R^2\}$ (*i.e.* D is the region consisting of points that are between distance r and R from the origin) with the constant density function $\delta(x, y, z) = \delta$. Express I_z in terms of m, r, and R, where m is the mass of D.

Solution. By definition, we have

$$\begin{aligned} I_z &= \iiint\limits_D (x^2+y^2)\cdot\delta(x,y,z)\,dV = \delta\iiint\limits_D (x^2+y^2)\,dV \\ &= \delta\cdot\int_0^{2\pi}\int_0^{\pi}\int_r^R (\rho\sin\varphi)^2\cdot\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \\ &= (2\pi\delta)\cdot\left[\int_0^{\pi}\sin^3\varphi\,d\varphi\right]\cdot\left[\int_r^R\rho^2\,d\rho\right] \\ &= \ldots = \frac{8\pi\delta}{15}(R^5-r^5). \qquad \text{[Exercise : Check this]} \end{aligned} \tag{31}$$

Now, we can check that the mass of D is

$$m = \iiint\limits_D \delta(x,y,z)\,dV = \delta\iiint\limits_D dV = \delta\cdot\text{Volume}(D) = \frac{4\pi\delta}{3}(R^3-r^3). \tag{32}$$

Hence we have

$$I_z = \frac{2m}{5}\cdot\frac{R^5-r^5}{R^3-r^3}. \tag{33}$$

□

Remark 10. There are two limiting cases here:

- if $r\to 0$, then our domain becomes a solid ball with inertia
$$I_z = \frac{2m}{5}R^2; \tag{34}$$
- if $r\to R$, then our domain becomes a (hollow) sphere made of an "infinitesimally thin" sheet. This has inertia
$$I_z = \lim_{r\to R}\frac{2m}{5}\cdot\frac{R^5-r^5}{R^3-r^3} = \frac{2m}{3}\cdot\frac{5R^4}{3R^2} = \frac{2m}{3}R^2. \tag{35}$$

We conclude that the inertia of the *hollow sphere* is greater than that of the *solid ball* (assuming the same uniformly distributed mass m).

Remark 11. For physical objects rotating about an axis, the amount of torque required to cause angular acceleration is proportional to the inertia about that axis. That is, inertia measures "how hard an object is to rotate". One can run physical experiments (or simulations) showing that the *hollow sphere* will take longer than the *solid ball* to roll down an incline (assuming they both have the same mass and radius and that they roll without slipping).

0.2 Change of Variables

0.2.1 Review in 1 Dimension

Recall that in the 1-dimensional case we can make the substitution $x=g(u)$ and have

$$\int_{g(a)}^{g(b)} f(x)\,dx = \int_a^b f\big(g(u)\big)\cdot g'(u)\,du. \tag{36}$$

(We are reversing the usual roles of x and u to end up with "nicer" forumlas in the end.)

By Taylor expansion (assuming g is "nice"), we can write

$$g(u_0+\Delta u) = g(u_0)+g'(u_0)\cdot\Delta u+\ldots \implies \Delta x = g(u_0+\Delta u)-g(u_0) = g'(u_0)\cdot\Delta u+\ldots \tag{37}$$

and so an infinitesimal change Δu at u_0 induces a change $\Delta x = g'(u_0)\cdot\Delta u$ at $x_0=g(u_0)$ up to first order, which is one way to interpret the expression $dx = g'(u)\cdot du$. Another way to say this is that $g'(u)=\frac{dx}{du}$ is the ratio of *lengths* under the change of coordinates $x=g(u)$.

There is a slight subtlety going on when we perform a change of variables. Consider the following example:

Example 12. Using the substitution $x = 2u$, we can write

$$\int_{-1}^{1} 8u^2\,du = \int_{-1}^{1} (2u)^2 \cdot 2\,du = \int_{-2}^{2} x^2\,dx = \ldots = \frac{16}{3} \tag{38}$$

Alternatively, we could substitute $x = -2u$ and write

$$\int_{-1}^{1} 8u^2\,du = \int_{-1}^{1} -(-2u)^2 \cdot (-2)\,dx = \int_{2}^{-2} -x^2\,dx = \ldots = \frac{16}{3}. \tag{39}$$

Both these substitutions are doing the same thing: we replace the relevant expression with x and change the bounds of integration to reflect the image of x, and ultimately, they give the same solution. One thing to note, however, is that one of them flips the bounds of integration. A change of variables that flips the bounds is called *orientation-reversing*, whereas one that does not is called *orientation-preserving*.

In 1 dimension, a change of variables $x = g(u)$ is *orientation-preserving* if $\frac{dx}{du} > 0$ everywhere (in its domain) and it is *orientation-reversing* if $\frac{dx}{du} < 0$ everywhere.

Notice that both the maps

$$u \mapsto 2u \text{ and } u \mapsto -2u \tag{40}$$

send the interval $[-1, 1]$ to $[-2, 2]$, but the second one flips the direction of the image.

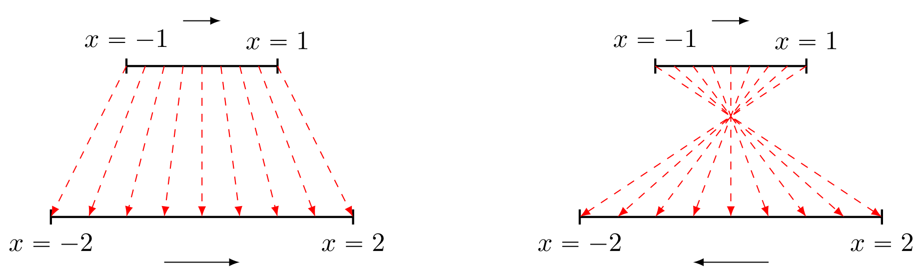

Figure 3: An orientation-preserving map and an orientation-reversing map.

If we were to integrate using the second change of variables, we then need something to reflect that we should be going in the "opposite direction", which is why the bounds of integration get flipped.

Ultimately, what this means is that we can write a 1-dimensional change of variables in terms of sets as follows: If $x = g(u)$, then

$$\int_{g(R)} f(x)\,dx = \int_R f\big(g(u)\big) \cdot \left|\frac{dx}{du}\right| du. \tag{41}$$

If our change of variables was *orientation-preserving*, then the absolute value does not contribute anything to the calculation. If instead our change of variables was *orientation-reversing*, then we get a factor of -1 from the absolute value, and another factor of -1 since we would traverse the image in the "wrong direction". These factors cancel out to make the expression work out properly.

Remark 13. In 1 dimension, the convention is that intervals are oriented "from left to right".

0.2.2 The 2- and 3-Dimensional Cases

In 2 dimensions, we have the same idea, but instead of lengths, we want areas. Suppose we have a change of coordinates $\Phi(u, v) = (x, y)$ given by

$$\begin{cases} x = g(u, v), \\ y = h(u, v). \end{cases} \tag{42}$$

Here $\Phi\colon R \to D$ can be thought of as a map from some domain $R \subseteq \mathbb{R}^2$ (with (u, v) coordinates) to another domain $D \subseteq \mathbb{R}^2$ (with (x, y) coordinates). What we want to find is the ratio of areas.

For now, let us suppose that Φ is a *diffeomorphism*, that is a smooth bijective map with smooth inverse. (A smooth function is a function which can be differentiated over and over again and still have continuous derivatives.) We can again Taylor expand and write

$$\begin{cases} g(u_0 + \Delta u, v_0 + \Delta v) = g(u_0, v_0) + \frac{\partial g}{\partial u}(u_0, v_0) \cdot \Delta u + \frac{\partial g}{\partial v}(u_0, v_0) \cdot \Delta v + \dots \\ h(u_0 + \Delta u, v_0 + \Delta v) = h(u_0, v_0) + \frac{\partial h}{\partial u}(u_0, v_0) \cdot \Delta u + \frac{\partial h}{\partial y}(u_0, v_0) \cdot \Delta v + \dots, \end{cases} \tag{43}$$

that is

$$\begin{cases} \Delta x = g(u_0 + \Delta u, v_0 + \Delta v) - g(u_0, v_0) = \frac{\partial g}{\partial u}(u_0, v_0) \cdot \Delta u + \frac{\partial g}{\partial v}(u_0, v_0) \cdot \Delta v + \dots \\ \Delta y = h(u_0 + \Delta u, v_0 + \Delta v) - h(u_0, v_0) = \frac{\partial h}{\partial u}(u_0, v_0) \cdot \Delta u + \frac{\partial h}{\partial v}(u_0, v_0) \cdot \Delta v + \dots \end{cases} \tag{44}$$

up to first order. We can represent this as a matrix

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial u}(u_0, v_0) & \frac{\partial g}{\partial v}(u_0, v_0) \\ \frac{\partial h}{\partial u}(u_0, v_0) & \frac{\partial h}{\partial v}(u_0, v_0) \end{bmatrix} \cdot \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} + \dots \tag{45}$$

and consider this as a linear map by ignoring the higher order terms.

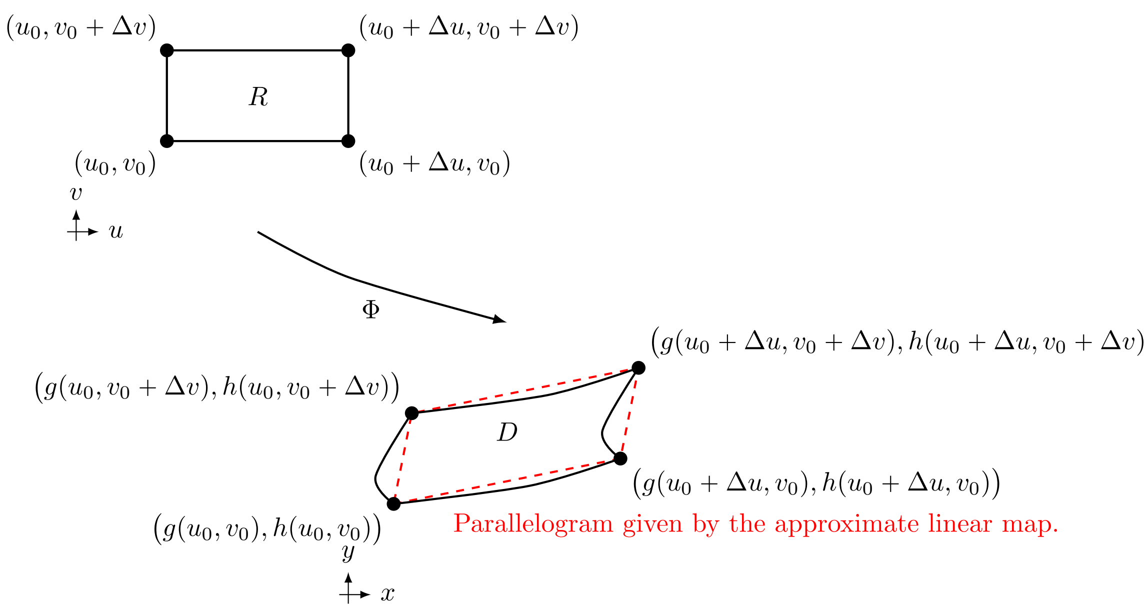

Figure 4: A change of variables causes a change in infinitesimal areas.

Remark 14. Since everything here only involved first order estimates, we actually only need Φ and Φ^{-1} to be C^1 (differentiable with continuous derivative) instead of smooth.

By linear algebra, the area of the approximate parallelogram after the transformation is

$$\text{Area}_{xy} = (\text{Area}_{uv}) \cdot \left| \det \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{bmatrix} \right|. \tag{46}$$

Definition 15 (2-Dimensional Jacobian)**.** The *Jacobian* (or *Jacobian determinant*) $J(u, v)$ of a change of coordinates $\Phi(u, v) = (x, y)$ given by

$$\begin{cases} x = g(u, v), \\ y = h(u, v) \end{cases} \tag{47}$$

is the determinant

$$J(u,v) = \det \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}. \tag{48}$$

The Jacobian is also sometimes denoted by

$$\frac{\partial(x,y)}{\partial(u,v)}. \tag{49}$$

This gives the 2-dimensional change of variables formula

$$\begin{aligned} \iint\limits_{D=\Phi(R)} f(x,y)\,dx\,dy &= \iint\limits_{R} f\big(x(u,v), y(u,v)\big) \cdot \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right| du\,dv \\ &= \iint\limits_{R} f\big(x(u,v), y(u,v)\big) \cdot |J(u,v)|\,du\,dv \\ &= \iint\limits_{R} f\big(x(u,v), y(u,v)\big) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv. \end{aligned} \tag{50}$$

Remark 16. In 2 dimensions, sets are usually oriented counter-clockwise (that is why we counter-clockwise angles are considered positive).

Example 17 (Polar Coordinates Revisited)**.** Consider polar coordinates in 2 dimensions. We have the change of coordinates

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta \end{cases} \tag{51}$$

and so the Jacobian $J(r,\theta)$ is

$$J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r. \tag{52}$$

This means that

$$\iint\limits_{D=\Phi(R)} f(x,y)\,dx\,dy = \int_R f(r,\theta) \cdot r\,dr\,d\theta, \tag{53}$$

which is the formula that we had from before.

Theorem 18 (2-Dimensional Change of Variables)**.** *Let $\Phi\colon (u,v) \mapsto (x,y)$ be a C^1 bijection with C^1 inverse, mapping a (closed and bounded) region R (in the uv-plane) to a (closed and bounded) region $D = \Phi(R)$ (in the xy-plane, except possibly at the boundary). If $f(x,y)$ is continuous on D, then*

$$\iint\limits_{D=\Phi(R)} f(x,y)\,dx\,dy = \iint\limits_{R} f \circ \Phi(u,v) \cdot \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv. \tag{54}$$

Remark 19. The conditions on the change of variables Φ guarantees that the Jacobian

$$|J(u,v)| \neq 0. \tag{55}$$

The idea is the same for 3-dimensional integrals. Let

$$\begin{aligned} \Phi\colon R &\to D \\ (u,v,w) &\mapsto (x,y,z) \end{aligned} \tag{56}$$

be a C^1 bijective map with C^1 inverse Φ^{-1} with

$$\begin{cases} x = g(u,v,w), \\ y = h(u,v,w), \\ z = k(u,v,w). \end{cases} \tag{57}$$

The 3-dimensional *Jacobian* is given by

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}. \tag{58}$$

Theorem 20 (3-Dimensional Change of Variables)**.** *Under similar conditions to Theorem 18, we have*

$$\iiint\limits_{D=\Phi(R)} f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{R} f \circ \Phi(u,v,w) \cdot \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

Remark 21. In 3 dimensions, sets are usually oriented using the *right-hand rule.*

Remark 22. The Chain Rule works in multiple dimensions as well. In 2 dimensions, we have

$$\frac{\partial(x,y)}{\partial(s,t)} \cdot \frac{\partial(s,t)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(u,v)} \tag{59}$$

and

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}. \tag{60}$$

In 3 dimensions, we similarly have

$$\frac{\partial(x,y,z)}{\partial(s,t,r)} \cdot \frac{\partial(s,t,r)}{\partial(u,v,w)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \tag{61}$$

and

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}}. \tag{62}$$

Example 23. Compute

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy. \tag{63}$$

Solution. We see that the inner limits of integration are the lines

$$x = \frac{y}{2} \longleftrightarrow 2x - y = 0 \tag{64}$$

and

$$x = \frac{y}{2} + 1 \longleftrightarrow 2x - y = 2. \tag{65}$$

We can use the change of variables

$$\begin{cases} u = 2x - y, \\ v = y \end{cases} \longleftrightarrow \begin{cases} x = \frac{1}{2}u + \frac{1}{2}v, \\ y = v. \end{cases} \tag{66}$$

Checking bounds on integration, we have

$$\begin{cases} x = \frac{y}{2}, \\ x = \frac{y}{2} + 1, \\ y = 0, \\ y = 4 \end{cases} \longleftrightarrow \begin{cases} u = 0, \\ u = 2, \\ v = 0, \\ v = 4. \end{cases} \tag{67}$$

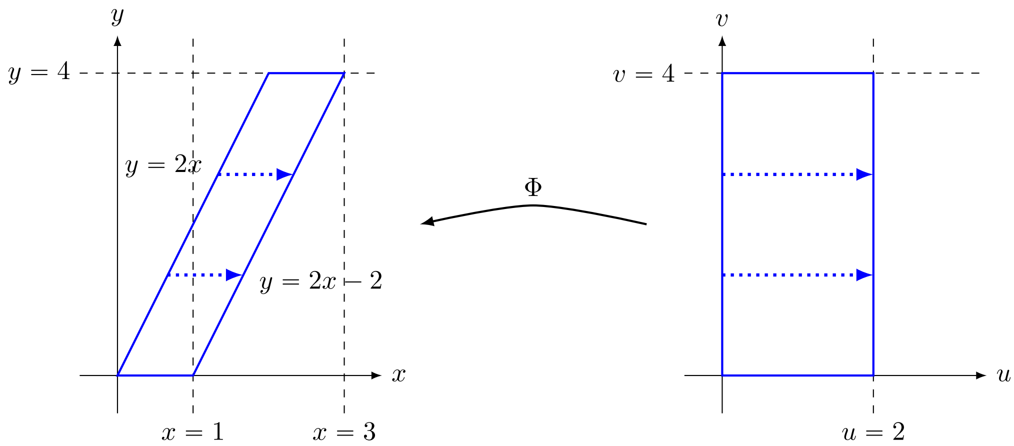

Figure 5: A change of coordinates.

The Jacobian in this case is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \frac{1}{2}. \tag{68}$$

Using the formula, we get

$$\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\, dx\, dy = \int_0^4 \int_0^2 \frac{u}{2} \cdot \left|\frac{1}{2}\right| du\, dv = \ldots = 2. \tag{69}$$

□

(End of Lecture 7 – Sep 29)