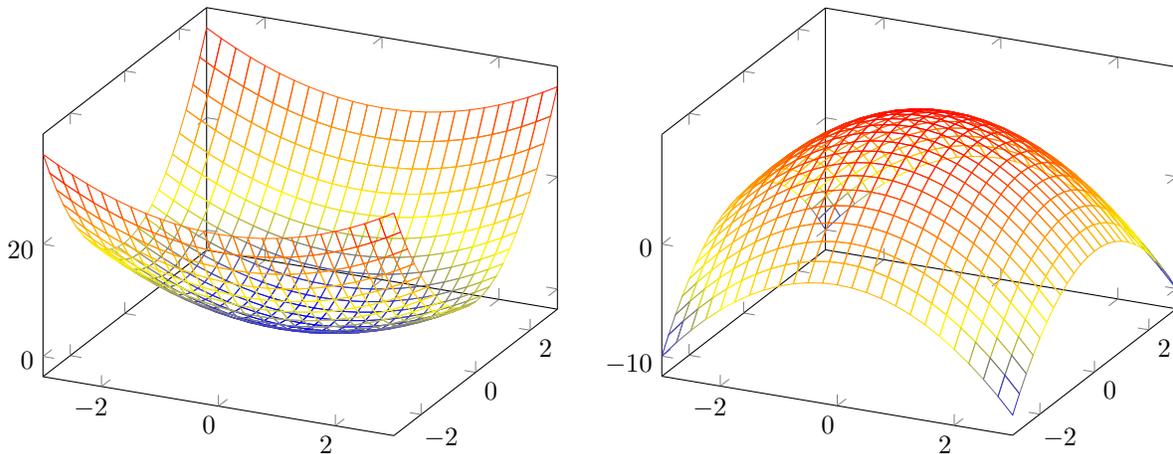# MATH2020A Lecture 6 Notes

Caleb Suan

Lat time, we looked at an example:

Example 1 (Volume Enclosed by Two Surfaces)**.** Find the volume of the region enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution. We can plot both surfaces to get an idea of how they look.

Figure 1: The surfaces $z = x^2 + 3y^2$ (left) and $z = 8 - x^2 - y^2$ (right).

We can first check where these surfaces intersect. Setting them equal to each other, we get the expression

$$x^2 + 3y^2 = 8 - x^2 - y^2 \implies \frac{x^2}{4} + \frac{y^2}{2} = 1, \tag{1}$$

which is an ellipse. This will enclose the entire region that we are investigating.

We see that our region can be expressed as

$$R = \left\{ (x, y, z) \in \mathbb{R}^3 \mid -2 \leqslant x \leqslant -2, -\sqrt{2 - \frac{x^2}{2}} \leqslant y \leqslant \sqrt{2 - \frac{x^2}{2}}, x^2 + 3y^2 \leqslant z \leqslant 8 - x^2 - y^2 \right\} \tag{2}$$

Applying Fubini's Theorem, we get the integral

$$\text{Volume} = \int_{-2}^{2} \int_{-\sqrt{2 - \frac{x^2}{2}}}^{\sqrt{2 - \frac{x^2}{2}}} \int_{x^2 + 3y^2}^{8 - x^2 - y^2} 1 \, dz \, dy \, dx$$

$$= \ldots = \int_{-2}^{2} \frac{16}{3} \left(2 - \frac{x^2}{2} \right) dx = \ldots = 8\sqrt{2}\pi. \qquad [\textcolor{magenta}{\textbf{Exercise}} : \text{Check this}]$$

$$\square$$

The main ideas here are about determining which surface is "above" and which is "below" and also where this is happening. In the above case, the area enclosed by the (projected) intersecting curve gave the region where $z = 8 - x^2 - y^2$ was "above" and where $z = x^2 + 3y^2$ was "below".

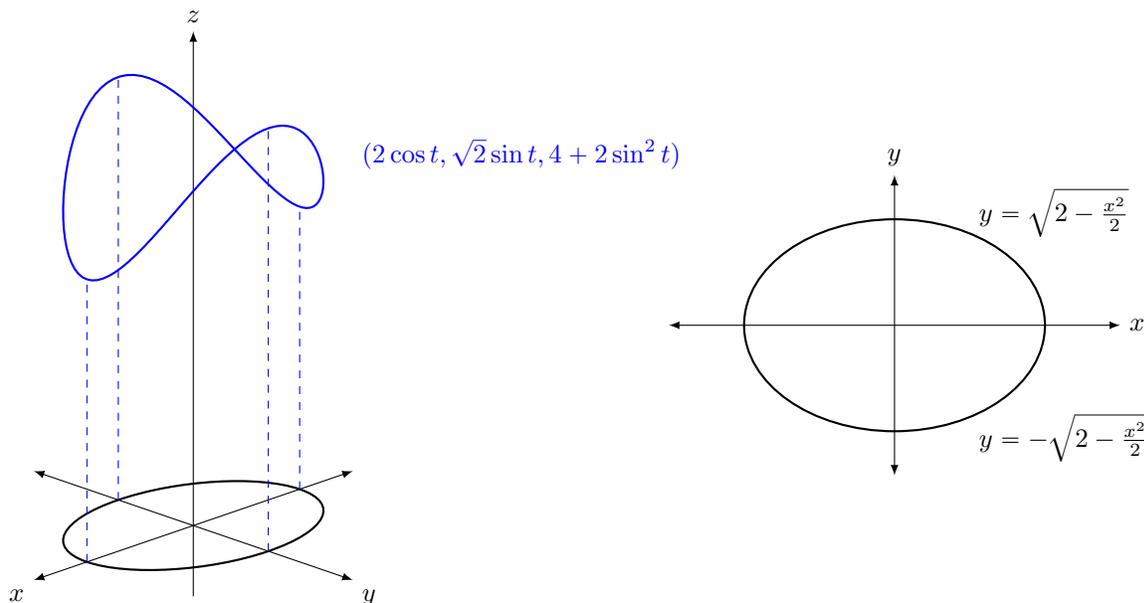**Remark 2.** One can actually show that the curve that our surfaces intersect can be parameterized by
$$x = 2\cos t, \quad y = \sqrt{2}\sin t, \quad z = 4 + 2\sin^2 t. \tag{3}$$
This can be done by parameterizing the ellipse
$$x = 2\cos t, \quad y = \sqrt{2}\sin t, \tag{4}$$
and then noticing that along our parameterization, we have
$$x^2 + 2y^2 = 4 \implies z = x^2 + 3y^2 = 4 + y^2 = 4 + 2\sin^2 t. \tag{5}$$

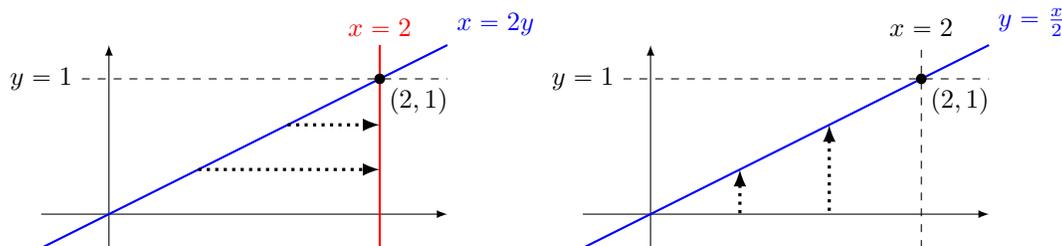**Figure 2:** The curve where the surfaces intersect and its projection to the xy-plane.

Example 3. Evaluate
$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz. \tag{6}$$

Solution. We compute directly
$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz = \int_0^4 \frac{2}{\sqrt{z}}\left[\int_0^1 \int_{2y}^2 \cos(x^2)\, dx\, dy\right] dz = \left[\int_0^4 \frac{2}{\sqrt{z}}\, dz\right] \cdot \left[\int_0^1 \int_{2y}^2 \cos(x^2)\, dx\, dy\right]. \tag{7}$$

The final integral can just be thought of as a regular 2-dimensional integral.


Figure 3: Computing the 2-dimensional integral.

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Using Fubini's Theorem, we get

$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz = \int_0^4 \frac{2}{\sqrt{z}}\left[\int_0^1 \int_{2y}^2 \cos(x^2)\, dx\, dy\right] dz$$

$$= \left[\int_0^4 \frac{2}{\sqrt{z}}\, dz\right] \cdot \left[\int_0^1 \int_{2y}^2 \cos(x^2)\, dx\, dy\right]$$

$$= \left[4\sqrt{z}\,\Big|_{z=0}^{z=4}\right] \cdot \left[\int_0^2 \int_0^{\frac{x}{2}} \cos(x^2)\, dy\, dx\right]$$

$$= \ldots = 8 \cdot \int_0^2 \frac{x}{2}\cos(x^2)\, dx$$

$$= \ldots = 2\sin(x^2)\,\Big|_{x=0}^{x=2} = 2\sin 4. \qquad\qquad [\textbf{Exercise} : \text{Check this}]$$

$\square$

From this example, we note that as long as they are independent from one another (remember to watch the bounds), we can separate the individual integrals from a triple integral. Also, after reducing to a double integral, we can use the same techniques as we did before, such as swapping the order of integration.

Example 4 (Volume of a Cone). Let B be a "nice" subset of $\mathbb{R}^2$. Let C be the cone in $\mathbb{R}^3$ with base B on the xy-plane and vertex $(0,0,h)$ with $h > 0$.
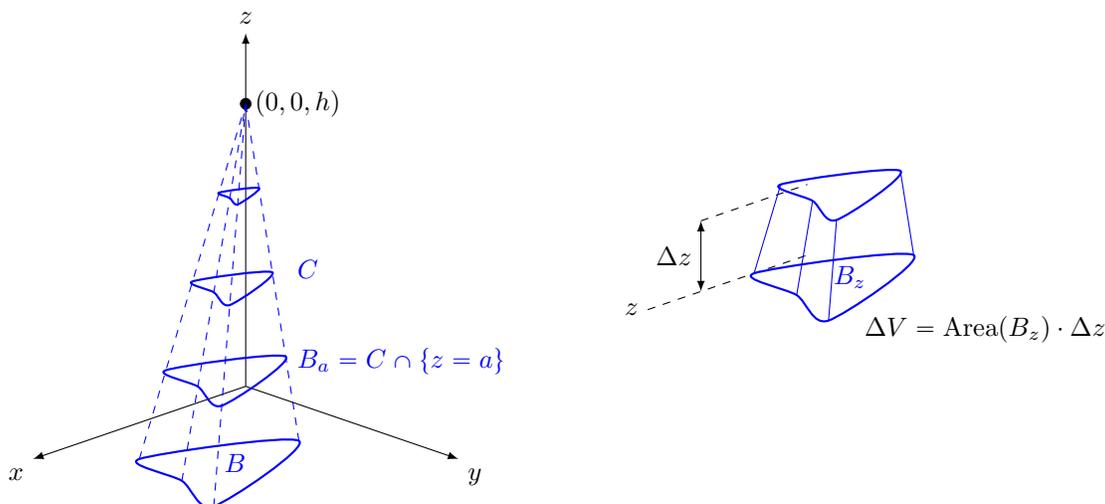
Figure 4: The cone C with base B and vertex $(0,0,h)$ and a horizontal slice B_z of the cone C.

What is the volume of C?

Using the idea of Riemann sums, we can add up horizontal layers of the cone and get

$$\text{Volume}(C) = \int_0^h \text{Area}(B_z)\, dz. \qquad\qquad (8)$$

The ratio of heights of B_z and the base B is $\frac{h-z}{h} = 1 - \frac{z}{h}$. By similarity, the ratio of areas must be

$$\frac{\text{Area}(B_z)}{\text{Area}(B)} = \left(1 - \frac{z}{h}\right)^2. \qquad\qquad (9)$$

This means that the volume is

$$\text{Volume}(C) = \int_0^h \left(1 - \frac{z}{h}\right)^2 \cdot \text{Area}(B)\, dz$$

$$= \text{Area}(B) \cdot \int_0^h \left(1 - \frac{z}{h}\right)^2 dz = \ldots = \frac{h}{3} \cdot \text{Area}(B). \qquad [\textbf{Exercise}: \text{Check this}] \qquad (10)$$

Remark 5. This is in line with the regular volume formula for a straight circular based cone

$$V = \frac{1}{3}\pi r^2 h. \qquad (11)$$

0.0.1 Alternative Coordinate Systems

In 2-dimensions, it was sometimes useful to write things in terms of polar coordinates. We will do a similar thing here and look at two other coordinate systems: cylindrical coordinates and spherical coordinates.

Cylindrical Coordinates The first coordinate system in 3-dimensions that we will consider are *cylindrical coordinates* (r, θ, z). This essentially converts the xy-plane into polar coordinates while leaving the z-coordinate unchanged. These are given by the change of coordinates formulae

$$\begin{cases} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z, \end{cases} \longleftrightarrow \begin{cases} r &= \sqrt{x^2 + y^2}, \\ \theta &= \arctan\left(\frac{y}{x}\right), \\ z &= z. \end{cases} \qquad (12)$$

As before, we need to be careful with the arctan function since we may have to add an additional angle depending on which quadrant the point is in.
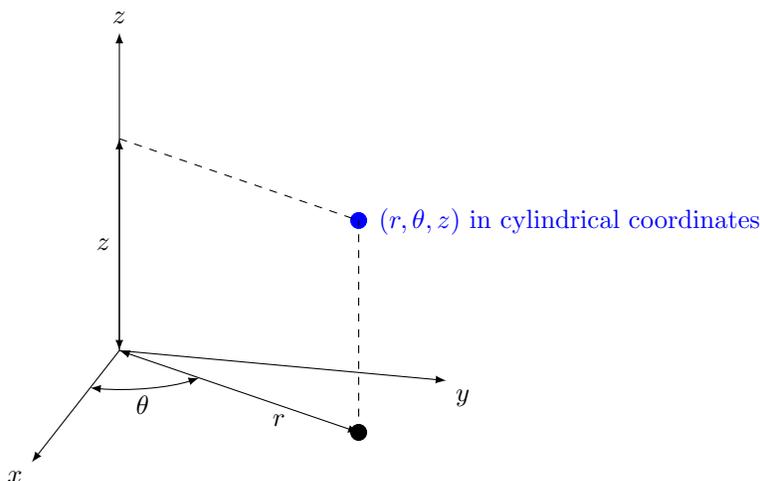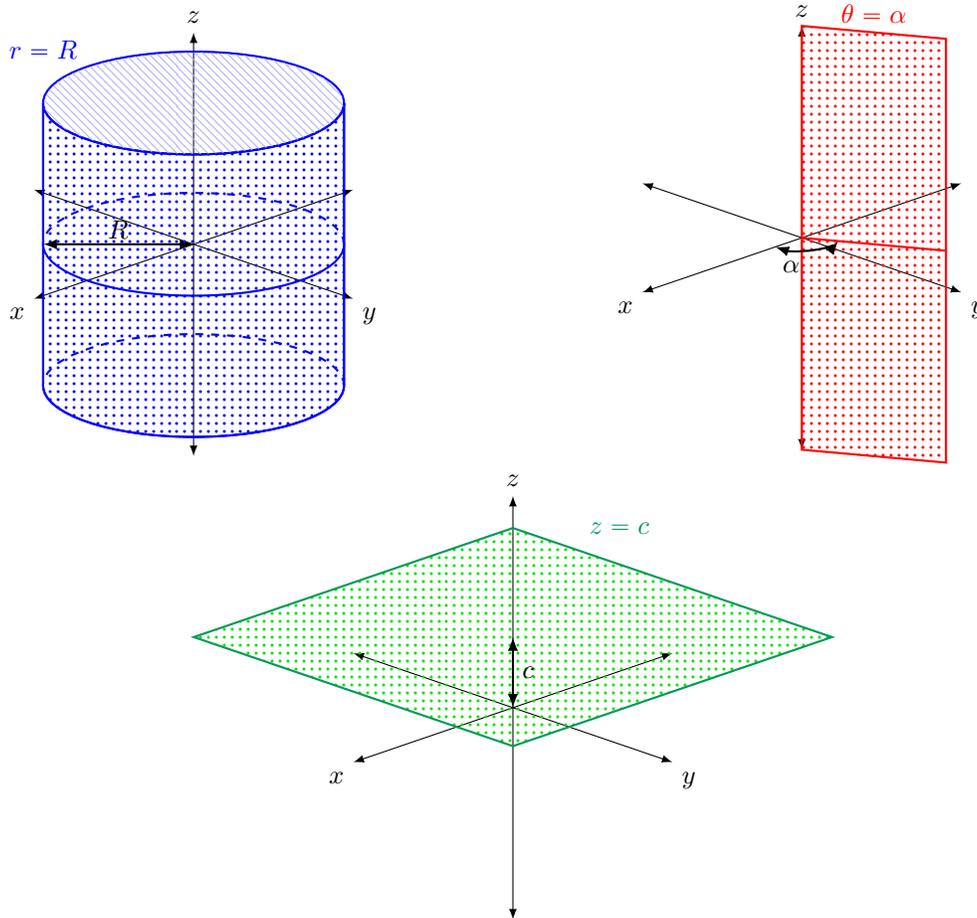

Figure 5: The cylindrical coordinate system.

Remark 6 (Names in Cylindrical Coordinates)**.** As in polar coordinates, r is sometimes referred to as the *radial distance* and θ is sometimes referred to as the *angular coordinate* or *azimuth*. The z-coordinate is usually called the *height* or *axial coordinate*.

Remark 7 (Level Sets in Cylindrical Coordinates)**.** Holding the different coordinates constant describes different geometric objects:

- the set $\{r = R\}$ describes a cylinder,
- the set $\{\theta = \alpha\}$ describes a vertical half-place,
- the set $\{z = c\}$ describes a horizontal plane (just like in Cartesian coordinates).

Figure 6: Level sets created by holding coordinates constant in cylindrical coordinates.

Remark 8 (Cylindrical Coordinates in Other Directions)**.** We can define cylindrical coordinates in other directions by choosing which coordinate stays unchanged. For example

$$\begin{cases} x & = x, \\ y & = r\cos\theta, \\ z & = r\sin\theta. \end{cases} \tag{13}$$

This has the effect of aligning the cylinder with one of the other axes (and may also change the "default" direction of angles).

We need to determine the volume element in these coordinates. Since cylindrical coordinates are effectively just polar coordinates with an additional (orthogonal) direction, we can take the area element $r\,dr\,d\theta$ and multiply by dz to get

$$dV = r\,dr\,d\theta\,dz. \tag{14}$$

Of course, if the relevant conditions are satisfied, we can exchange the order of integaration using Fubini's Theorem.

Example 9 (Volume of an Ice Cream Cone)**.** Find the volume of the ice cream cone I as shown in the figure.

Solution. We can recognize that this is a solid obtained by rotating around the z-axis from $\theta = 0$ to $\theta = 2\pi$, and so by keeping θ fixed, we obtain a uniform slice of the solid.
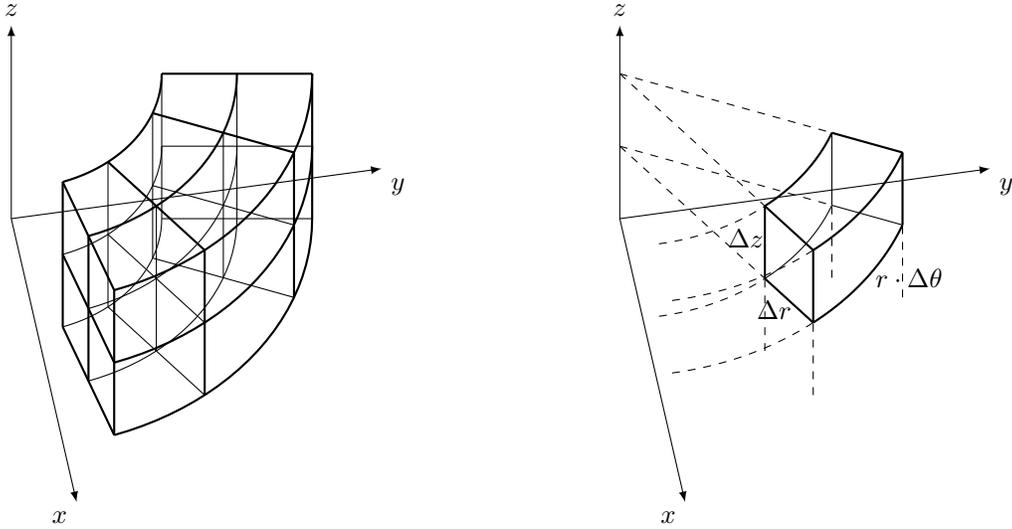
Figure 7: A partition in cylindrical coordinates and an infinitesimal volume element.
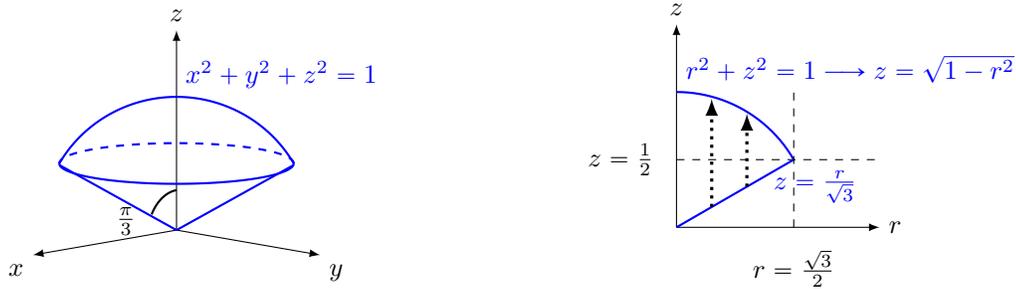
Figure 8: An ice cream cone and its cross-section.

By Fubini's Theorem, the volume of the ice cream cone is given by the integral

$$\text{Volume}(I) = \iiint_I dV = \int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}} \int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta$$

$$= 2\pi \cdot \int_0^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{r}{\sqrt{3}} \right) \cdot r \, dz$$

$$= 2\pi \cdot \left[-\frac{1}{3}\sqrt{1-r^2}^{\frac{3}{2}} - \frac{r^3}{3\sqrt{3}} \Big|_{r=0}^{r=\frac{\sqrt{3}}{2}} \right]$$

$$= 2\pi \cdot \left(-\frac{1}{3} \cdot \frac{1}{8} - \frac{1}{8} + \frac{1}{3} \right) = \frac{\pi}{3}. \tag{15}$$

$\square$

Remark 10. Note that we still need to have the factor of r in the integral from the change of coordinates even though we are calculating a volume. It is there because it is part of the volume element in cylindrical coordinates.

Spherical Coordinates   Another way to describe points in $\mathbb{R}^3$ is through *spherical coordinates* (ρ, φ, θ). This keeps the same idea as cylindrical coordinates for the xy-plane, but now first uses an angle φ (called the *inclination* or *polar angle*) to describe the "height" of a point. The change of coordinates formulae in

this case is

$$\begin{cases} x &= \rho \sin \varphi \cos \theta, \\ y &= \rho \sin \varphi \sin \theta, \\ z &= \rho \cos \varphi, \end{cases} \longleftrightarrow \begin{cases} \rho &= \sqrt{x^2 + y^2 + z^2}, \\ \varphi &= \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right), \\ \theta &= \arctan\left(\frac{y}{x}\right). \end{cases} \tag{16}$$

Once again, we need to be careful with the arctan function since we may have to add an additional angle depending on which quadrant the point is in.

Remark 11 (Different Conventions for Spherical Coordinates)**.** There are different conventions in use on how to order the angles in spherical coordinates (so make sure you check if you are using other sources). Some common notations are

$$(\underbrace{\rho}_{\text{radial}}, \underbrace{\theta}_{\text{azimuthal}}, \underbrace{\varphi}_{\text{polar}}), \tag{17}$$

and

$$(\underbrace{\rho}_{\text{radial}}, \underbrace{\theta}_{\text{polar}}, \underbrace{\varphi}_{\text{azimuthal}}). \tag{18}$$

Some sources also define the polar angle differently, measuring it "upwards" from the xy-plane instead.

For these notes, we will be using the notation in the textbook (Thomas' Calculus [15th Edition])

$$(\underbrace{\rho}_{\text{radial}}, \underbrace{\varphi}_{\text{polar}}, \underbrace{\theta}_{\text{azimuthal}}) \tag{19}$$

which are given by the change of coordinate formulae above. With this convention, we generally restrict the angles φ and θ to the ranges $0 \leqslant \varphi \leqslant \pi$ and $0 \leqslant \theta \leqslant 2\pi$ respectively.
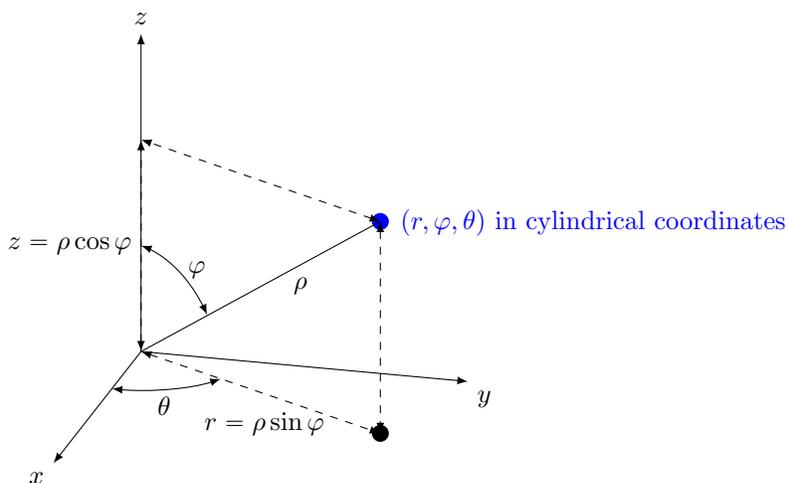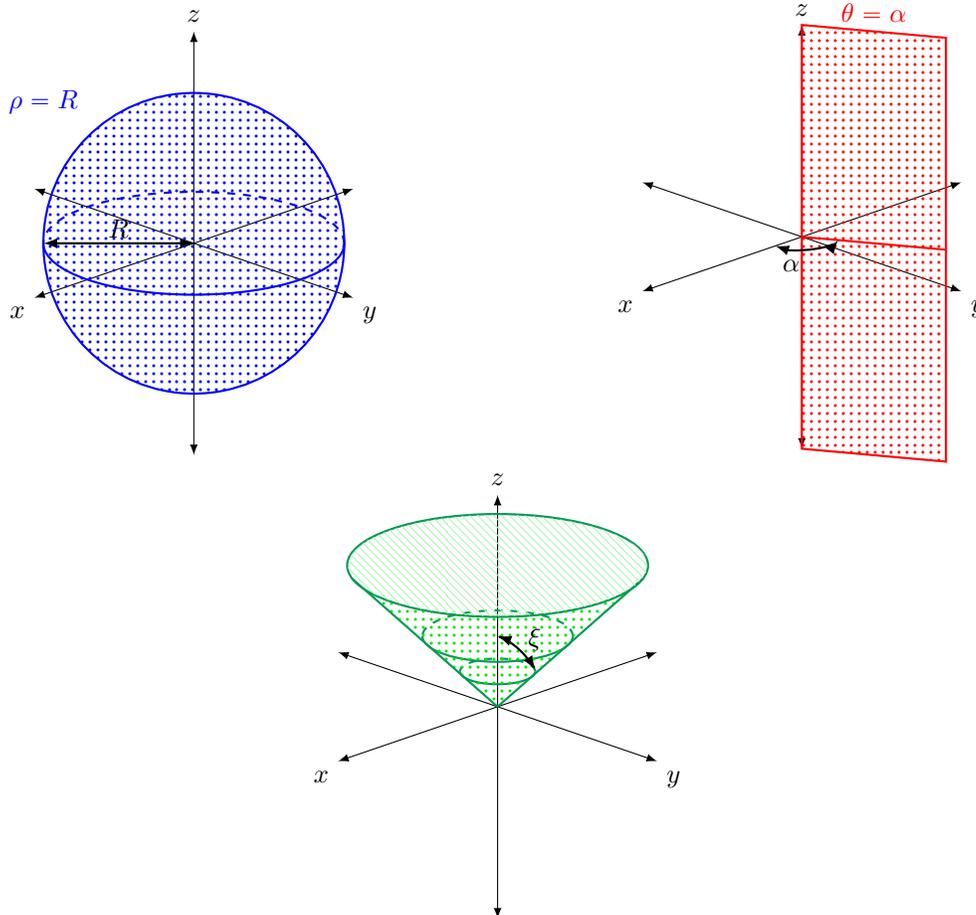

Figure 9: The spherical coordinate system.

Remark 12 (Converting between Different Coordinate Systems)**.** We have the following relations between the three systems:

$$\begin{array}{ccccc} x & = & r \cos \theta & = & \rho \sin \varphi \cos \theta \\ y & = & r \sin \theta & = & \rho \sin \varphi \sin \theta \\ z & = & z & = & \rho \cos \varphi \\ \text{Cartesian} & \longleftrightarrow & \text{Cylindrical} & \longleftrightarrow & \text{Spherical} \end{array} \tag{20}$$

Remark 13 (Level Sets in Spherical Coordinates)**.** Holding the different coordinates constant describes different geometric objects:

- the set $\{\rho = R\}$ describes a cylinder,
- the set $\{\varphi = \xi\}$ describes,
  - the positive z-axis if $\xi = 0$;
  - the negative z-axis if $\xi = \pi$;
  - the xy-plane if $\xi = \frac{\pi}{2}$;
  - a cone otherwise (the cone opens upwards when $\xi < \frac{\pi}{2}$, and opens downwards when $\xi > \frac{\pi}{2}$).
- the set $\{\theta = \alpha\}$ describes a vertical half-place.

**Figure 10:** Level sets created by holding coordinates constant in spherical coordinates.

Remark 14. The level sets of φ are all mathematical cones (even in the special cases $\varphi = 0$, $\varphi = \frac{\pi}{2}$, and $\varphi = \pi$). These are defined as spaces that are closed under addition and multiplication by positive scalars.

The volume element in spherical coordinates is given by

$$dV = \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta. \tag{21}$$

Example 15 (Converting from Cartesian to Spherical Coordinates). Convert the following into spherical coordinates:

- the sphere $x^2 + y^2 + (z-1)^2 = 1$,
- the cone $z = -\sqrt{x^2 + y^2}$.
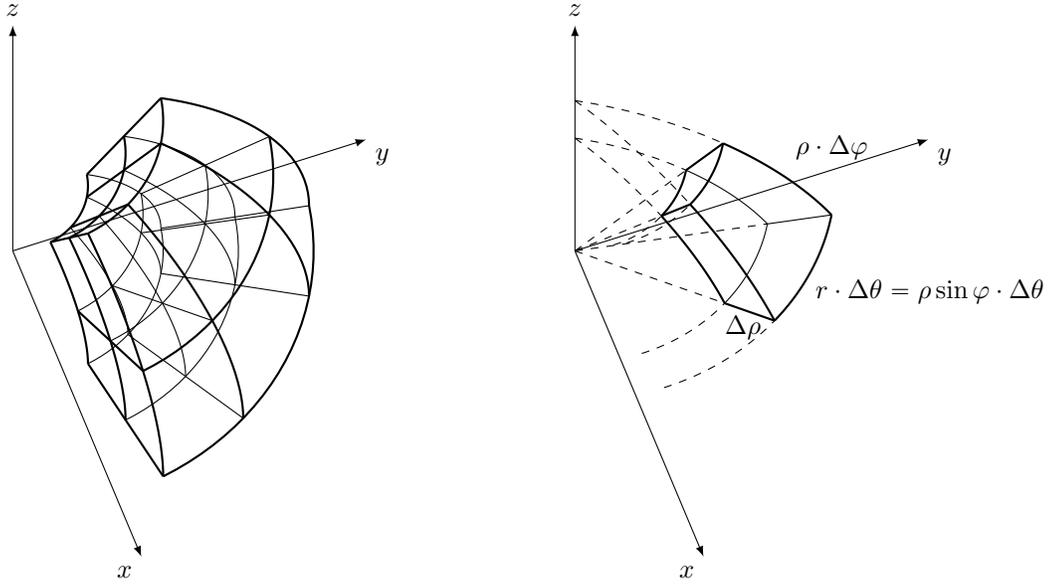
Figure 11: A partition in spherical coordinates and an infinitesimal volume element.

Solution.   • We begin by substituting using the change of coordinates formulae

$$x^2 + y^2 + z^2 = 1 \implies \rho^2 \sin^2 \varphi \cos^2 \theta + \rho^2 \sin^2 \varphi \sin^2 \theta + (\rho \cos \varphi - 1)^2 = 1$$

$$\implies \rho^2 \sin^2 \varphi + \rho^2 \cos^2 \varphi - 2\rho \cos \varphi + 1 = 1$$

$$\implies \rho^2 = 2\rho \cos \varphi$$

$$\implies \rho = 2 \cos \varphi. \tag{22}$$

• Again, by substituting, we get

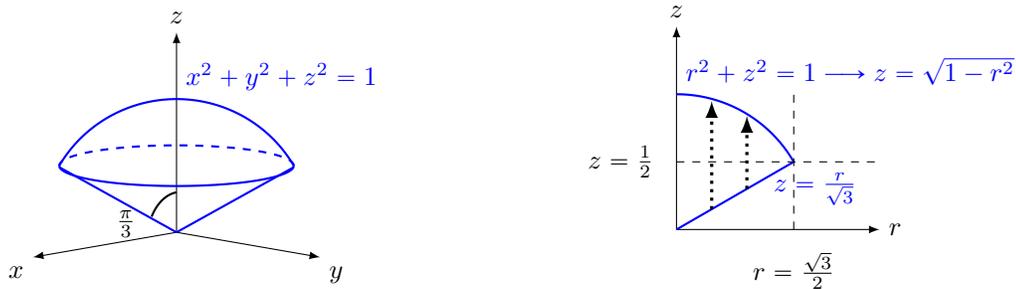$$z = -\sqrt{x^2 + y^2} \implies \rho \cos \varphi = \rho \sin \varphi. \tag{23}$$

When $\rho = 0$, we get the origin $(0, 0, 0)$ and when $\rho \neq 0$, we get

$$\cos \varphi = -\sin \varphi \implies \varphi = \frac{3\pi}{4}. \tag{24}$$

(Recall that we choose $0 \leqslant \varphi \leqslant \pi$.)

$\square$

Example 16 (Example 9 Revisited)**.** Find the volume of the ice cream cone I as shown in the figure using spherical coordinates.

Figure 12: An ice cream cone and its cross-section.

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Solution. From the figure, we can see that the ice cream cone I is just the intersection of the unit sphere $x^2 + y^2 + z^2 = 1$ and the cone $\varphi \leqslant \frac{\pi}{3}$. Writing everything in spherical coordinates shows that

$$I = \{(\rho, \varphi, \theta) \in \mathbb{R}^3 \mid 0 \leqslant \rho \leqslant 1, 0 \leqslant \varphi \leqslant, 0 \leqslant \theta \leqslant 2\pi\}. \tag{25}$$

To find the volume, we compute the integral (remember to include the $\rho^2 \sin\varphi$ in the spherical volume element)

$$\begin{aligned}
\text{Volume}(I) &= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 1 \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \\
&= \left[\int_0^{2\pi} d\theta \right] \cdot \left[\int_0^{\frac{\pi}{3}} \sin\varphi \, d\varphi \right] \cdot \left[\int_0^1 \rho^2 \, d\rho \right] \\
&= (2\pi) \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{3}\right) = \frac{\pi}{3}.
\end{aligned} \tag{26}$$

Of course, this aligns with our previous calculation. $\square$

_____(**End of Lecture 6 − Sep 25**)