MATH2020A Lecture 5 Notes

Caleb Suan

Last time, we discussed polar coordinates, given by the change of variables

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2+y^2}, \\ \theta = \arctan(\frac{y}{x}) + \text{corrective constant depending on quadrant.} \end{cases} \tag{1}$$

We also noted that the area element in polar coordinates is given by

$$dx\,dy \longleftrightarrow r\,dr\,d\theta. \tag{2}$$

We now continue with some examples of integrals in polar coordinates:

Example 1 (Converting an Integral from Cartesian to Polar Coordinates)**.** Convert

$$\int_1^2 \int_0^{\sqrt{2x-x^2}} y\,dy\,dx \tag{3}$$

into polar coordinates.

Solution. We start by drawing a figure to help us visualize the domain D. To convert this into polar coordinates, we can notice that the angles in the region range from 0 to $\frac{\pi}{4}$. Then using the change of coordinates formulae, we get

$$x = 1 \longleftrightarrow r\cos\theta = 1, \tag{4}$$

$$y = \sqrt{2x-x^2} \longleftrightarrow r\sin\theta = \sqrt{2r\cos\theta - r^2\cos^2\theta}. \tag{5}$$

We now want to turn these into expressions for r in terms of θ. The first can be rewritten as

$$r = \sec\theta, \tag{6}$$

while for the second one, we have

$$\begin{aligned} r\sin\theta &= \sqrt{2r\cos\theta - r^2\cos^2\theta} \\ r^2\sin^2\theta &= 2r\cos\theta - r^2\cos^2\theta \\ r^2 &= 2r\cos\theta \\ r &= 2\cos\theta. \end{aligned}$$

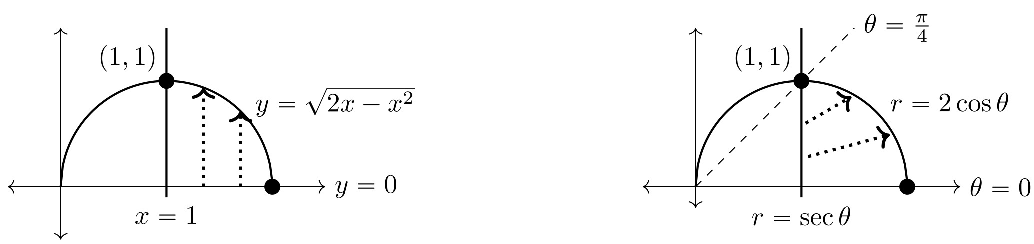

Figure 1: Converting D from a Cartesian domain to a polar one.

Since $y = r\sin\theta$, we finally get that the (double) integral can be written as

$$\int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\cos\theta} r^2 \sin\theta \, dr \, d\theta. \tag{7}$$

Again, we note that since we wrote our domain as a Type I Polar domain, we need to integrate over r first, then θ. □

Example 2 (Area Enclosed by a Lemniscate)**.** Find the area enclosed by the curve $r^2 = 4\cos(2\theta)$.

Solution. To make things easier, we first note that by the symmetry of the domain, we only need to work with the first quadrant ($0 \leqslant \theta \leqslant \frac{\pi}{2}$).

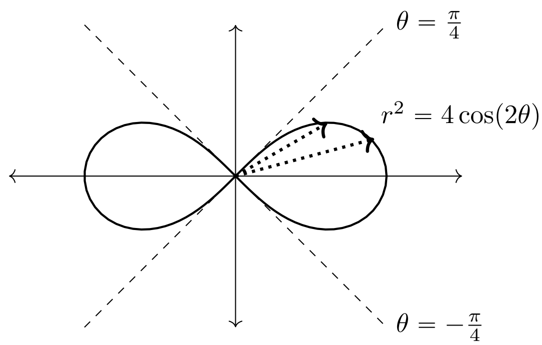

Figure 2: The curve $r^2 = 4\cos(2\theta)$ (also known as the Lemniscate of Bernoulli).

We can restrict the values of θ further by nothing that there are no solutions to $r^2 = 4\cos(2\theta)$ when $\theta > \frac{\pi}{4}$. Ultimately, this suggests the integral

$$\begin{aligned}
\text{Area} &= 4 \cdot \int_0^{\frac{\pi}{4}} \int_0^{2\sqrt{\cos(2\theta)}} r \, dr \, d\theta \\
&= 4 \cdot \int_0^{\frac{\pi}{4}} \left. \frac{r^2}{2} \right|_{r=0}^{r=2\sqrt{\cos\theta}} d\theta \\
&= 4 \cdot \int_0^{\frac{\pi}{4}} 2\cos 2\theta \, d\theta = \ldots = 4. \qquad \text{[Exercise : Check this.]}
\end{aligned} \tag{8}$$

□

Remark 3. In this case, r is not exactly a function of θ and should be regarded as a *level set*:

- there is no solution when $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ and when $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$,
- in terms of Cartesian coordinates, we can rewrite the expression as

  $$r^2 = 4\cos(2\theta) \longleftrightarrow (x^2 + y^2)^2 = 4(x^2 - y^2) \tag{9}$$

  and so the lemniscate is given by the *level set*

  $$F(x, y) = (x^2 + y^2)^2 - 4(x^2 - y^2) = 0. \tag{10}$$

  [**Exercise** : Check this]

  This has a *critical point* at the origin $(0, 0)$ since $\overrightarrow{\nabla} F(0, 0) = \overrightarrow{0}$. (This means that we cannot apply the Implicit Function Theorem at the *critical point* $(0, 0)$.)

Example 4 (Area Bounded between Two Curves)**.** Integrate the function $f(x, y) = \frac{1}{\sqrt{x^2+y^2}}$ over the region R inside the cardioid

$$r = 1 + \cos\theta, \tag{11}$$

and outside the circle

$$r = 1. \tag{12}$$

Solution. We first find the points of intersection for the curves. Solving

$$1 + \cos\theta = 1, \tag{13}$$

we get that $\theta = \pm\frac{\pi}{2}$ which corresponds to the points $(-\frac{\pi}{2}, 1)$ and $(\frac{\pi}{2}, 1)$.

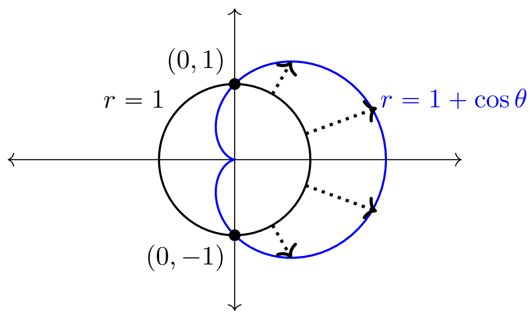

Figure 3: The cardioid $r = 1 + \cos\theta$ and the unit circle $r = 1$.

We then get the integral

$$\iint\limits_R f(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_1^{1+\cos\theta} \frac{1}{r}\cdot r\,dr\,d\theta = \ldots = 2. \qquad [\textbf{Exercise} : \text{Check this}] \tag{14}$$

□

Example 5 (Average Height of a Hemisphere)**.** Let $z = \sqrt{a^2 - x^2 - y^2}$ be a function defined on $R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leqslant a^2\}$. The *graph* of z is the (upper) hemisphere of radius a. Find the average height of the hemisphere.

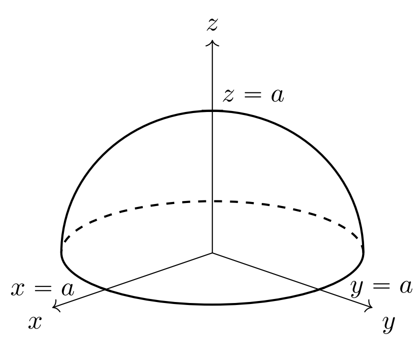

Figure 4: A plot of the upper hemisphere $f(x,y) = \sqrt{a^2 - x^2 - y^2}$.

Solution. We have the integral

$$\begin{aligned} \text{Average Height} &= \frac{1}{\text{Area}(R)} \iint\limits_R z\,dA \\ &= \frac{1}{\pi a^2} \int_0^{2\pi} \int_0^a \sqrt{a^2 - r^2}\cdot r\,dr\,d\theta = \ldots = \frac{2a}{3}. \qquad [\textbf{Exercise} : \text{Check this}] \end{aligned} \tag{15}$$

□

Sometimes, considering 2 dimensions can be helpful for even 1-dimensional problems.

Example 6 (Gaussian Integral)**.** Compute $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ (recall that this is an improper integral since the domain of integration is unbounded).

Solution. Consider the 2-dimensional integral

$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy \tag{16}$$

(this is also an improper integral since the domain of integral is unbounded).

We can calculate this integral as the limit of what happens if we integrated over larger and larger circles. [**Exercise** : How does this compare with 1-dimensional improper integrals?]

$$\begin{aligned}\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy &= \lim_{\rho\to\infty} \iint_{\{x^2+y^2\leqslant\rho\}} e^{-x^2-y^2}\,dx\,dy \\ &= \lim_{\rho\to\infty} \int_0^{2\pi}\int_0^{\rho} e^{-r^2}\cdot r\,dr\,d\theta \\ &= \lim_{\rho\to\infty} \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_{r=0}^{r=\rho}\,d\theta \\ &= \lim_{\rho\to\infty} \pi(1-e^{-\rho^2}) = \pi. \end{aligned} \tag{17}$$

In the above, we set the region of integration to be the circle of radius ρ and took the limit as $\rho\to\infty$ to eventually encompass the entire plane.

On the other hand, we can use squares of larger and larger side length instead.

$$\begin{aligned}\iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dx\,dy &= \lim_{k\to\infty}\int_{-k}^{k}\int_{-k}^{k} e^{-x^2-y^2}\,dx\,dy \\ &= \lim_{k\to\infty}\left(\int_{-k}^{k} e^{-x^2}\,dx\right)\cdot\left(\int_{-k}^{k} e^{-y^2}\,dy\right) \\ &= \lim_{k\to\infty}\left(\int_{-k}^{k} e^{-x^2}\,dx\right)^2 \\ &= \left(\lim_{k\to\infty}\int_{-k}^{k} e^{-x^2}\,dx\right)^2 \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2. \end{aligned} \tag{18}$$

Here, we integrated over the square $[-k,k]\times[-k,k]$. Taking the limit as $k\to\infty$ eventually covers all of $\mathbb{R}^2$.

Comparing these, we conclude that

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}. \tag{19}$$

□

Remark 7. Note that we used two different limiting processes. Why should these be equal? [Hints: Remember that $e^{-x^2} > 0$ and consider Figure 5 below.]

0.1 Triple Integrals

We now move on to 3-dimensional integrals. Pretty much everything that we have covered carries over into this setting.

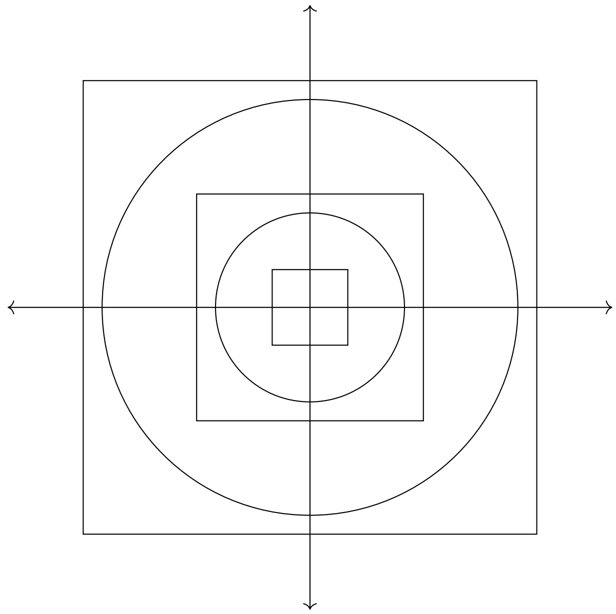

Figure 5: We can interweave the different regions of integration together.

Definition 8 (Triple Integral over a Rectangular Box)**.** Let $f(x, y, z)$ be a function defined on a closed (and bounded) rectangular box

$$B = [a, b] \times [c, d] \times [r, s]. \tag{20}$$

Then the (triple) integral of f over the box B is

$$\iiint\limits_B f(x, y, z)\, dV = \lim_{\|P\|\to 0} \sum_{k=1}^{m} \sum_{l=1}^{n} \sum_{p=1}^{q} f(x_{k,l,p}, y_{k,l,p}, z_{k,l,p}) \cdot \Delta V_{k,l,p}, \tag{21}$$

if the limit exists.

As before

- $P = P_1 \times P_2 \times P_3$ is a subdivision of B into sub-rectangular boxes by partitions P_1, P_2, and P_3 of $[a, b]$, $[c, d]$, and $[r, s]$ respectively. Also $\|P\| = \max(\|P_1\|, \|P_2\|, \|P_3\|)$;
- $(x_{k,l,p}, y_{k,l,p}, z_{k,l,p})$ is an arbitrary point in the sub-rectangular box $B_{k,l,p}$;
- $\Delta V_{k,l,p}$ is the volume of the box $B_{k,l,p}$ and is given by

$$\Delta V_{k,l,p} = \Delta x_k \cdot \Delta y_l \cdot \Delta z_p. \tag{22}$$

Fubini's Theorem naturally extends here and allows us to pick the order in which we integrate over the variables x, y, and z:

Theorem 9 (Fubini's Theorem (Triple Integrals, First Form))**.** *If $f(x, y, z)$ is a* continuous *(in fact* absolutely integrable *is sufficient) function on the box $B = [a, b] \times [c, d] \times [r, s]$, then*

$$\begin{aligned}\iiint\limits_B f(x, y, z)\, dV &= \int_r^s \left[\int_c^d \left[\int_a^b f(x, y, z)\, dx \right] dy \right] dz \\ &= \underbrace{\cdots}_{\text{Any order of } x,\ y,\ \text{and } z \text{ works}} = \int_a^b \left[\int_c^d \left[\int_r^s f(x, y, z)\, dz \right] dy \right] dx.\end{aligned}$$

Similarly, to integrate over a general (and bounded) region D, we simply need to pick a sufficiently large box $B \supseteq D$ and extend f appropriately.

Definition 10 (Triple Integral on a General Domain)**.** Let D be a closed (and bounded) domain in $\mathbb{R}^3$ and let $f(x, y, z)$ be a function defined on D. We define the (triple) integral of f over D to be

$$\iiint_D f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV, \tag{23}$$

where $B \supseteq D$ is a closed (and bounded) rectangular box containing D and

$$F(x,y,z) = \begin{cases} f(x,y,z), & \text{if } (x,y,z) \in D, \\ 0, & \text{if } (x,y,z) \in B \setminus D. \end{cases} \tag{24}$$

Like in the 2-dimensional case, this is well-defined and does not depend on the choice of box B.

0.1.1 Special Domains

In 2-dimensions, we had special domains where one variable ranged within a certain interval, and the other lied between functions dependent on the first. In 3-dimensions, we take this one step further, and allow the final variable to vary between functions dependent on the other two variables:

- (Type I) $D = \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in R_1, u_1(x,y) \leqslant z \leqslant u_2(x,y)\}$, where R_1 is a closed (and bounded) *plane region* (that is, a 2-dimensional special domain) and u_1 and u_2 are *continuous* functions on R_1 with $u_1(x,y) \leqslant u_2(x,y)$,
- (Type II) $D = \{(x,y,z) \in \mathbb{R}^3 \mid (x,z) \in R_2, v_1(x,z) \leqslant y \leqslant u_2(x,z)\}$, where R_2 is a closed (and bounded) *plane region* and v_1 and v_2 are *continuous* functions on R_2 with $v_1(x,z) \leqslant v_2(x,z)$,
- (Type III) $D = \{(x,y,z) \in \mathbb{R}^3 \mid (y,z) \in R_3, w_1(y,z) \leqslant x \leqslant w_2(y,z)\}$, where R_3 is a closed (and bounded) *plane region* and w_1 and w_2 are *continuous* functions on R_3 with $w_1(y,z) \leqslant w_2(y,z)$.

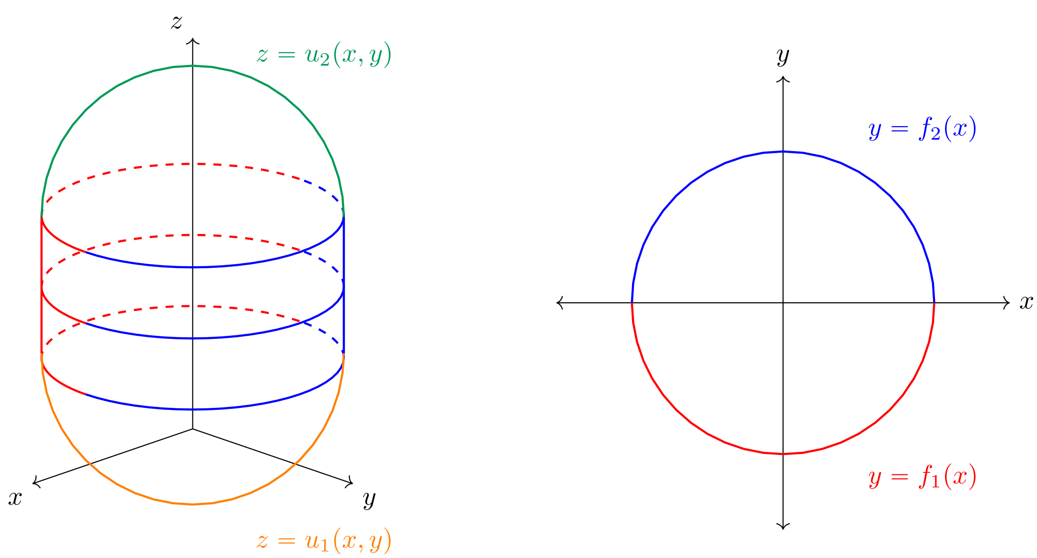

Figure 6: A 3-dimensional Type I domain and the plane domain it is built from.

On these domains, we get the expected extension of Fubini's Theorem:

Theorem 11 (Fubini's Theorem (Triple Integrals, Stronger Form))**.** *Let $f(x,y,z)$ be a continuous (or at least* absolutely integrable*) function on a domain D. If D is of Type I (as above), then*

$$\iiint_D f(x,y,z)\,dV = \iint_{R_1} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dx\,dy. \tag{25}$$

Similar results hold for domains of Types II and III.

Using Fubini's Theorem (for double integrals), we can write such triple integrals as a sequence of three iterated (single) integrals. For example, if

$$R = \{(x, y, z) \in \mathbb{R}^3 \mid a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x), u_1(x, y) \leqslant z \leqslant u_2(x, y)\}, \tag{26}$$

then

$$\iiint_R f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\, dy\, dx. \tag{27}$$

Proposition 12 (Properties of Triple Integrals)**.** *The properties of integrals (linearity, domination, and additivity over disjoint domains) hold for triple integrals over general "good" domains in* $\mathbb{R}^3$*.*

Example 13. Find the volume of domain D in the first octant ($x \geqslant 0, y \geqslant 0, z \geqslant 0$) bounded by the plane $6x + 4y + 3z = 12$.

Solution. By projecting to the xy-plane (setting $z = 0$), we see that the plane $6x + 4y + 3z = 12$ intersects it along the line $6x + 4y = 12$ or equivalently $y = 3 - \frac{3}{2}x$. We can also check that our x-values in our domain range from $x = 0$ to $x = 2$.

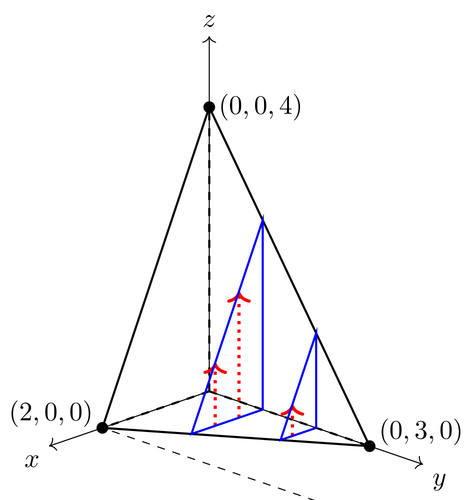

Figure 7: The domain of integration D.

This implies that our domain D can be written as

$$D = \left\{(x, y, z) \in \mathbb{R}^3 \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 3 - \frac{3}{2}x, 0 \leqslant z \leqslant 4 - 2x - \frac{4}{3}y\right\} \tag{28}$$

and our corresponding integral should be

$$\begin{aligned} \text{Volume} &= \iint_D 1\, dV = \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{4-2x-\frac{4}{3}y} 1\, dz\, dy\, dx \\ &= \int_0^2 \int_0^{3-\frac{3}{2}x} \left(4 - 2x - \frac{4}{3}y\right) dy\, dx \\ &= \ldots = \int_0^2 \left(\frac{3}{2}x^2 - 6x + 6\right) dx = \ldots = 4. \end{aligned}$$

[**Exercise** : Check this]

□

Remember that the way we have set up our integral sums over z first, then over y, then finally over x.

[**Exercise** : Generalize the previous example to the domain R in the first octant bounded by the place $ax + by + cz = d$ for $a, b, c, d > 0$.]

Example 14 (Volume of an Ellipsoid)**.** Find the volume of the ellipsoid D given by

$$D = \left\{(x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1\right\}, \quad (a, b, c > 0). \tag{29}$$

Solution. By symmetry of the ellipsoid, it suffices to consider only the first octant. Projecting onto the xy-plane ($z = 0$) we get the boundary curve

$$y = b\sqrt{1 - \frac{x^2}{a^2}}. \tag{30}$$

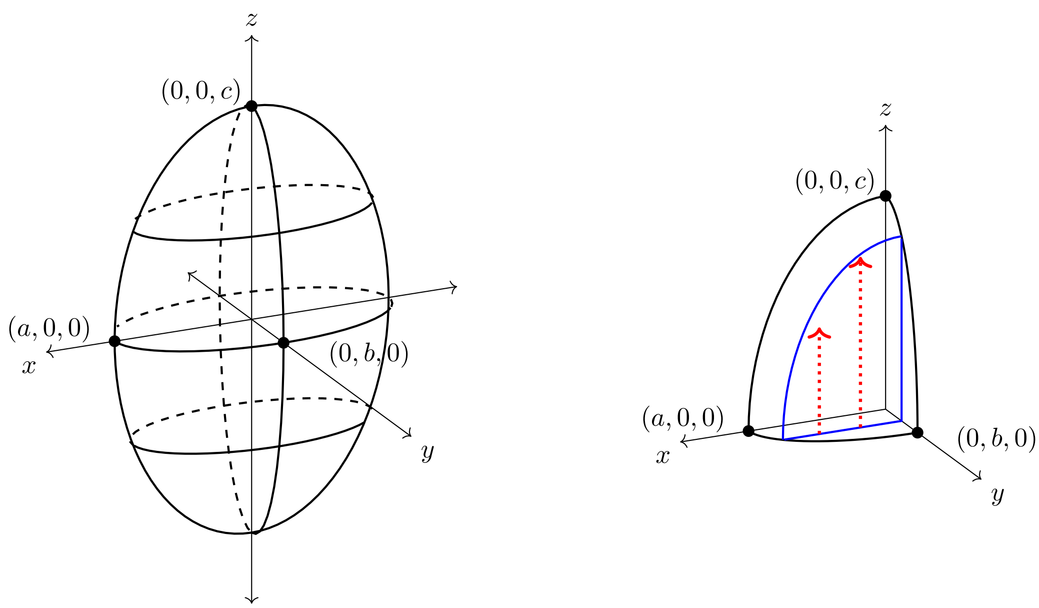

Figure 8: The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leqslant 1$ and its intersection with the first octant.

This implies that the volume should be

$$\text{Volume} = 8 \cdot \text{Volume in First Octant} = \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} 1 \, dz \, dy \, dx = \ldots = \frac{4\pi}{3} abc.$$

□

We won't bother too much with the actual calculation for now (though you may decide to do it as an optional **Exercise**). We will see a nicer way to tackle this integral later.

Example 15 (Volume Enclosed by Two Surfaces)**.** Find the volume of the region enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Solution. We can plot both surfaces to get an idea of how they look.

We can first check where these surfaces intersect. Setting them equal to each other, we get the expression

$$x^2 + 3y^2 = 8 - x^2 - y^2 \implies \frac{x^2}{4} + \frac{y^2}{2} = 1, \tag{31}$$

which is an ellipse. This will enclose the entire region that we are investigating.

We see that our region can be expressed as

$$R = \left\{(x, y, z) \in \mathbb{R}^3 \mid -2 \leqslant x \leqslant -2, -\sqrt{2 - \frac{x^2}{2}} \leqslant y \leqslant \sqrt{2 - \frac{x^2}{2}}, x^2 + 3y^2 \leqslant z \leqslant 8 - x^2 - y^2\right\} \tag{32}$$

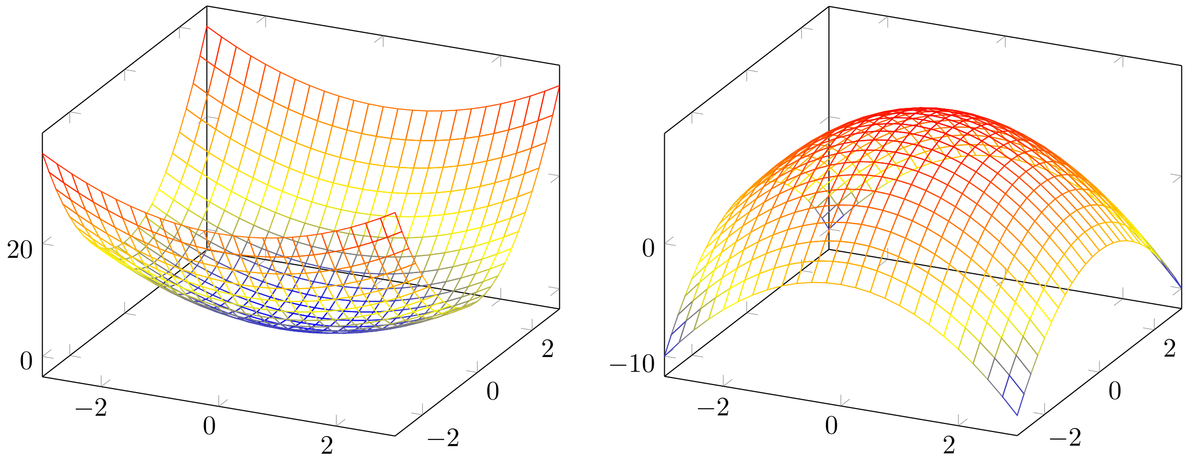

Figure 9: The surfaces $z = x^2 + 3y^2$ (left) and $z = 8 - x^2 - y^2$ (right).

Applying Fubini's Theorem, we get the integral

$$\begin{aligned}\text{Volume} &= \int_{-2}^{2} \int_{-\sqrt{2-\frac{x^2}{2}}}^{\sqrt{2-\frac{x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} 1\,dz\,dy\,dx \\ &= \ldots = \int_{-2}^{2} \frac{16}{3}\left(2 - \frac{x^2}{2}\right) dx = \ldots = 8\sqrt{2}\pi. \end{aligned}$$

[**Exercise** : Check this]

□

(End of Lecture 5 – Sep 22)