

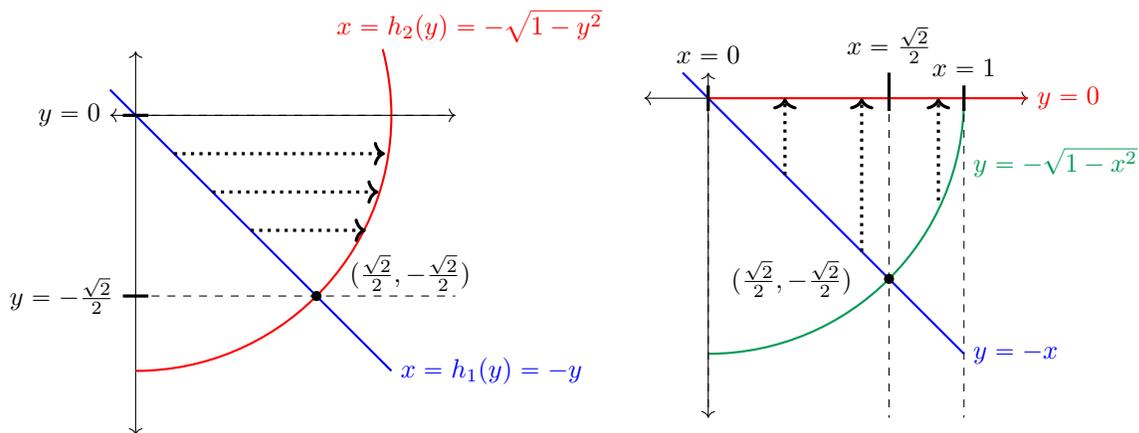
# MATH2020A Lecture 4 Notes

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Last time, we had an example where we wanted to integrate over the circular sector.

**Example 1.** Find  $\iint_D x \, dA$ , where  $D$  is the region in the right half-plane bounded by the lines  $y = 0$ ,  $x + y = 0$ , and the unit circle.

We ultimately tackled this by writing our region as either a Type II domain or a combination of two Type I domains.



**Figure 1:** The domain of integration  $D$  as a Type II domain (left) and as a combination of two Type I domains (right)

The first method resulted in the computation

$$\iint_D x \, dA = \int_{-\frac{\sqrt{2}}{2}}^0 \left[ \int_{-y}^{\sqrt{1-y^2}} x \, dx \right] dy = \int_{-\frac{\sqrt{2}}{2}}^0 \left( \frac{1}{2} - y^2 \right) dy = \dots = \frac{1}{3\sqrt{2}} \quad [\text{Exercise : Check this.}] \quad (1)$$

While the second gave us This gives the computation

$$\begin{aligned} \iint_D x \, dA &= \int_0^{\frac{\sqrt{2}}{2}} \left[ \int_{-x}^0 x \, dy \right] dx + \int_{\frac{\sqrt{2}}{2}}^1 \left[ \int_{-\sqrt{1-x^2}}^0 x \, dy \right] dx \\ &= \int_0^{\frac{\sqrt{2}}{2}} x^2 \, dx + \int_{\frac{\sqrt{2}}{2}}^1 x \cdot \sqrt{1-x^2} \, dx = \dots = \frac{1}{3\sqrt{2}} \quad [\text{Exercise : Check this.}] \quad (2) \end{aligned}$$

While certainly doable, some of the computations end up not being particular nice or clean, however, a change of perspective can make things easier.

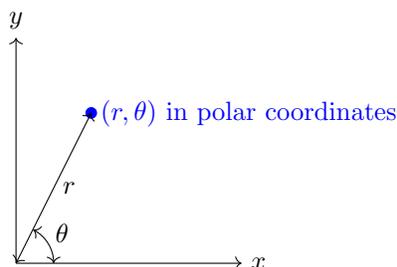
## 0.1 Double Integrals in Polar Coordinates

Recall that we can express points  $(x, y)$  in  $\mathbb{R}^2$  using polar coordinates  $(r, \theta)$ . To do this, we convert using the formulae

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta, \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{x^2 + y^2}, \\ \theta = \arctan\left(\frac{y}{x}\right). \end{cases} \quad (3)$$

This represents every point in the plane  $\mathbb{R}^2$  uniquely if we restrict  $\theta$  to the interval  $[0, 2\pi)$  (except for the origin).

**Remark 2.** We have to be slightly careful with using the arctan function to get the angle coordinate  $\theta$  since a correction might need to be added depending on which quadrant the point  $(x, y)$  lies in.



**Figure 2:** The polar coordinate system.

**Remark 3** (Names in Polar Coordinates). The coordinate  $r$  is sometimes referred to as the *radial distance* and  $\theta$  is sometimes referred to as the *angular coordinate* or *azimuth*.

How does using polar coordinates affect (double) integrals? Recall that in Cartesian coordinates  $(x, y)$  we started by defining (double) integrals on rectangles using the Riemann sum

$$\sum_{k=1}^m \sum_{l=1}^n f(x_{k,l}, y_{k,l}) \cdot \Delta A_{k,l}. \quad (4)$$

The main thing that happened was that we took an interval  $[a, b]$  for  $x$  and another interval  $[c, d]$  for  $y$  and got the set

$$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\}, \quad (5)$$

which was a rectangle that we later partitioned.

If we do this for polar coordinates, we no longer end up with a rectangle. Instead, if we consider the set

$$D = \{(r, \theta) \in \mathbb{R}^2 \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}, \quad (6)$$

we get part of a ring (also known as a sector of an annulus).

The Riemann sum formula picks a point in each of these sub-sectors and multiplies by the area of the sub-sector. As such, we need an expression for this area.

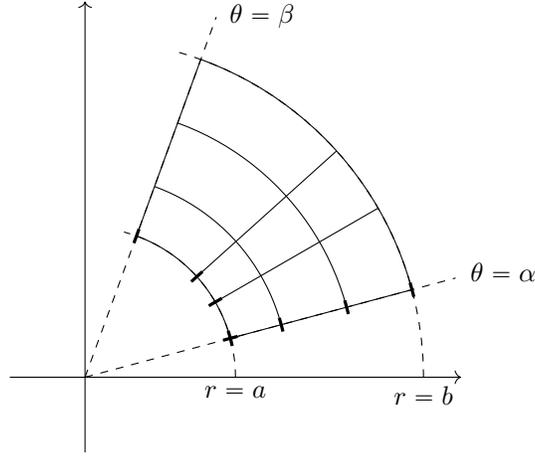
At very small scales (when  $r_{k-1}$  is very close to  $r_k$ ), this shape is approximately a rectangle. This gives the approximate area formula

$$\Delta A_{k,l} \approx (r_k \cdot \Delta \theta_l) \cdot \Delta r_k \quad (7)$$

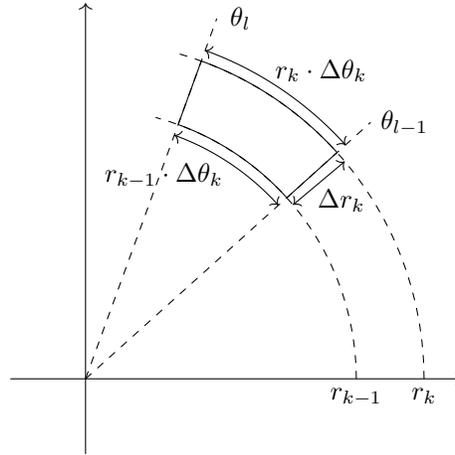
or more generally

$$\Delta A \approx r \cdot \Delta r \cdot \Delta \theta. \quad (8)$$

**Remark 4** (Area of an Infinitesimal Annular Sector). One can check [ [Exercise](#) ] that the actual area of



**Figure 3:** Partitioning up a sector of an annulus.



**Figure 4:** An approximate area formula.

the sector is given by

$$\begin{aligned}
 \frac{1}{2} \Delta \theta_l \cdot r_k^2 - \frac{1}{2} \Delta \theta_l \cdot r_{k-1}^2 &= \frac{1}{2} \Delta \theta_l \cdot (r_k - r_{k-1}) \cdot (r_k + r_{k-1}) \\
 &= \frac{1}{2} \Delta \theta_l \cdot \Delta r_k \cdot (r_k + r_{k-1}) \\
 &\approx r_k \cdot \Delta \theta_l \cdot \Delta r_k
 \end{aligned} \tag{9}$$

when  $r_{k-1}$  is very close to  $r_k$ , which gives the same approximate formula as above.

This means that our (double) integral should then look like

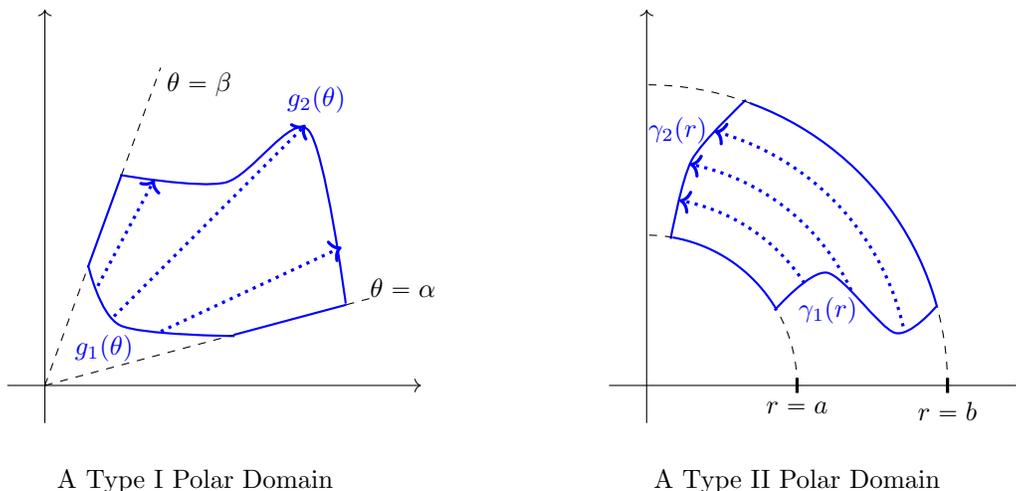
$$\iint_R f(x, y) dA = \iint_R f(x, y) dx dy = \iint_R f(r \cos \theta, r \sin \theta) r dr d\theta. \tag{10}$$

**Remark 5** (Jacobian Determinant). This extra factor of  $r$  that appears comes as a result of using a different coordinate system and is a special case of the general change of variable formula (to be covered later). It is in fact the *Jacobian determinant* of the change of variables:

$$r = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}. \tag{11}$$

After taking the change of coordinates into account, we get new special domains.

- (Type I Polar)  $D = \{(r, \theta) \in \mathbb{R}^2 \mid g_1(\theta) \leq r \leq g_2(\theta), \alpha \leq \theta \leq \beta\}$ , where  $g_1$  and  $g_2$  are *continuous* functions on  $[\alpha, \beta]$  with  $g_1(\theta) \leq g_2(\theta)$  for  $\theta \in [\alpha, \beta]$ .
- (Type II Polar)  $D = \{(r, \theta) \in \mathbb{R}^2 \mid a \leq r \leq b, \gamma_1(r) \leq \theta \leq \gamma_2(r)\}$ , where  $\gamma_1$  and  $\gamma_2$  are *continuous* functions on  $[a, b]$  with  $\alpha_1(r) \leq \alpha_2(r)$  for  $r \in [a, b]$ .



**Figure 5:** Special domains in  $\mathbb{R}^2$  using polar coordinates.

We have the following version of Fubini's Theorem:

**Theorem 6** (Fubini's Theorem (Polar Coordinates)). *Let  $f(r, \theta)$  be a continuous function on a closed (and bounded) domain  $D$ .*

1. If  $D$  is of Type I Polar as above, then

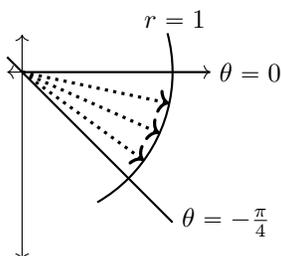
$$\iint_D f(x, y) dA = \int_{\alpha}^{\beta} \left[ \int_{g_1(\theta)}^{g_2(\theta)} f(r, \theta) r dr \right] d\theta, \quad (12)$$

2. If  $D$  is of Type II Polar as above, then

$$\iint_D f(x, y) dA = \int_a^b \left[ \int_{\gamma_1(r)}^{\gamma_2(r)} f(r, \theta) r d\theta \right] dr. \quad (13)$$

**Example 7** (Example 1 Revisited). Find  $\iint_D x dA$ , where  $D$  is the region in the right half-plane bounded by the lines  $y = 0$ ,  $x + y = 0$ , and the unit circle.

*Solution.* We can realize this domain as a special polar domain.



**Figure 6:** Writing  $R$  as a special polar domain.

In polar coordinates, the integrand is

$$x = r \cos \theta, \quad (14)$$

and so the integral becomes

$$\iint_D x \, dA = \int_{-\pi/4}^0 \left[ \int_0^1 r \cos \theta \cdot r \, dr \right] d\theta = \left[ \int_{-\pi/4}^0 \cos \theta \, d\theta \right] \cdot \left[ \int_0^1 r^2 \, dr \right] = \dots = \frac{1}{3\sqrt{2}}, \quad (15)$$

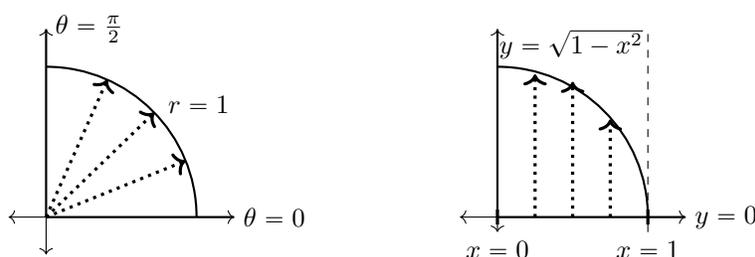
[ **Exercise** : Check this.] which aligns with our previous calculations.  $\square$

This calculation was easier than when we previously did it. Sometimes, it helps to convert integrals to polar coordinates (especially when dealing with circles, ellipses [kind of], or other conic sections). Conversely, there are situations where Cartesian coordinates become much more useful.

**Example 8** (Converting an Integral from Polar to Cartesian Coordinates). Convert

$$\int_0^{\pi/2} \int_0^1 r^3 \sin \theta \cos \theta \, dr \, d\theta \quad (16)$$

into Cartesian coordinates.



**Figure 7:** Converting  $D$  from a polar domain to a Cartesian one.

*Solution.* We first rewrite the expressions for the boundary curves in terms of  $x$  and  $y$ . This gives

$$\theta = \frac{\pi}{2} \longleftrightarrow x = 0 \quad (17)$$

$$r = 1 \longleftrightarrow y = \sqrt{1-x^2} \quad (18)$$

$$\theta = 0 \longleftrightarrow y = 0. \quad (19)$$

We note also that the unit circle gives the additional constraints that  $x \leq 1$  and  $y \leq 1$ .

Putting all this together, we get the domain

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq \sqrt{1-x^2}\}. \quad (20)$$

It remains to rewrite the integrand. Keeping in mind to leave out a factor of  $r$  for the change of coordinates, we get

$$\int_0^{\pi/2} \int_0^1 r^3 \sin \theta \cos \theta \, dr \, d\theta = \int_0^{\pi/2} \int_0^1 (r \sin \theta) \cdot (r \cos \theta) \cdot r \, dr \, d\theta = \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dy \, dx. \quad (21)$$

$\square$

Note that the order of integration matters in the final step since we decided to write the domain as a Type I domain. (Otherwise the bound involving  $\sqrt{1-x^2}$  would not make sense.)

(End of Lecture 4 – Sep 18)