MATH2020A Lecture 3 Notes

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Last time, we looked at two examples of non-integrable functions. We considered the function

$$f(x,y) = \begin{cases} 0, & \text{if both } x \text{ and } y \text{ are rational,} \\ 1, & \text{otherwise.} \end{cases} \tag{1}$$

over the rectangle $R = [0,1] \times [0,1]$ and also the function

$$f(x,y) = \begin{cases} \frac{1}{xy}, & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 0, & \text{if } x = 0 \text{ or } y = 0. \end{cases} \tag{2}$$

over the same rectangle.

The previous two examples show that we need certain conditions to ensure the *integrability* of a function over a closed (and bounded) rectangle.

Proposition 1. *Let $R = [a,b] \times [c,d]$ be a closed (and bounded) rectangle. If $f(x,y)$ is integrable on R, then f is bounded on R (i.e. there exists $M > 0$ such that $|f(x,y)| \leqslant M$ for all $(x,y) \in R$).*

Proof. Omitted (see Example 8 from last week for an idea of the proof). □

Proposition 2 (Conditions for Integrability)**.** *Let $R = [a,b] \times [c,d]$ be a closed (and bounded) rectangle. If a function $f(x,y)$ is* bounded *and* continuous *on R, then f is* integrable *on R.*

Proof. Omitted. (See the proof of the 1-variable case in MATH2060 for an idea of the proof.) □

In fact, Proposition 2 can be generalized to functions that are continuous on R except for on a "small" set. The precise concept here is called a *set of measure zero* (see MATH4050 Real Analysis). We get the following refinement:

Proposition 3 (Conditions for Integrability (Stronger Version))**.** *Let $R = [a,b] \times [c,d]$ be a closed (and bounded) rectangle. If a function $f(x,y)$ is* bounded *and* continuous *on R, except possibly at finitely many points or curves, then f is* integrable *on R.*

Proof. Omitted. □

We now state some properties of (double) integrals.

Proposition 4 (Properties of Double Integrals)**.** *Let $R = [a,b] \times [c,d]$ be a closed (and bounded) rectangle, let $f(x,y)$ and $g(x,y)$ be functions on R, and let $k \in \mathbb{R}$ be a constant.*

1. *(Linearity) If f and g are* integrable *over R, the $f \pm g$ and kf are also* integrable *over R.*

   In this case, we have

$$\iint\limits_R (f \pm g)\, dA = \iint\limits_R f\, dA + \iint\limits_R g\, dA, \tag{3}$$

and

$$\iint_R kf\,dA = k \cdot \iint_R f\,dA. \tag{4}$$

2. *(Domination) If $f \geqslant g$, then*

$$\iint_R f\,dA \geqslant \iint_R g\,dA. \tag{5}$$

In particular, when $g \equiv 0$, we see that if $f \geqslant 0$, then

$$\iint_R f\,dA \geqslant 0. \tag{6}$$

Proof. Omitted. (Clear from the properties of Riemann sums.) ☐

Remark 5. Point 1 of Proposition 4 implies that the set of integrable functions over a (fixed) closed (and bounded) rectangle R forms a *vector space over* $\mathbb{R}$ and that taking (double) integrals is a *linear operator*.

0.1 Double Integrals over General Domains

Instead of keeping the rectangle constant and adding functions, we can think about what happens when we add rectangles together. When we put two rectangles R and R' together, we can have different configurations:

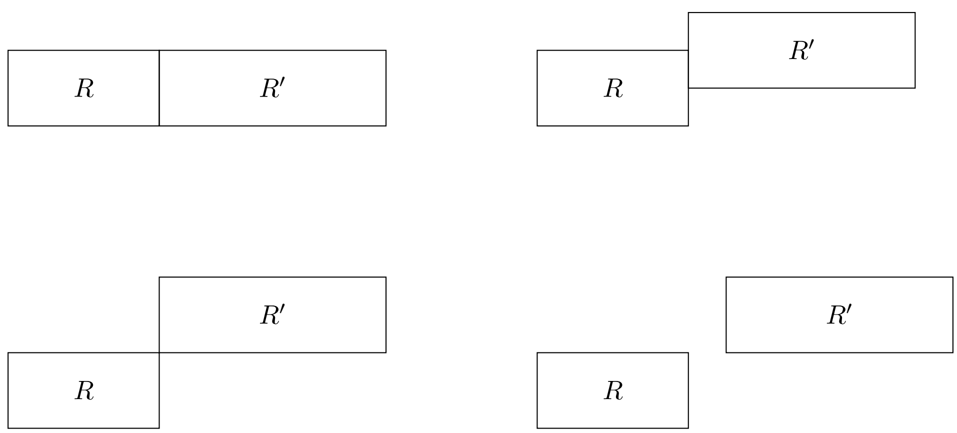


Figure 1: Various ways to combine two rectangles R and R' together.

We see that only in the first case do we get another rectangle. As such, with our tools so far we cannot define the (double) integral

$$\iint_{R \cup R'} f\,dA \tag{7}$$

in the other three cases. To get around this, we need to define (double) integrals over general domains in $\mathbb{R}^2$.

For a general closed (and bounded) domain R, we define the (double) integral by considering a larger rectangle that it fits in.

Definition 6 (Double Integral over General Domains)**.** Let D be a closed (and bounded) domain in $\mathbb{R}^2$ and $f(x, y)$ be a function defined on D. For any rectangle $R \supseteq D$, define

$$F(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \in D, \\ 0, & \text{if } (x, y) \in R \setminus D. \end{cases} \tag{8}$$

We define the *(double) integral of f over R* to be

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA. \tag{9}$$

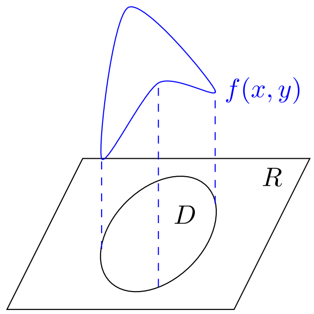


Figure 2: When integrating a function f over a domain D, we choose a rectangle R enclosing D.

Remark 7 (Well-Definition of the Double Integral over General Domains)**.** This definition is *well-defined* (*i.e.* it does not depend on our choice of rectangle R): If R' is another rectangle with $D \subseteq R'$ and we set

$$F'(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D, \\ 0, & \text{if } (x,y) \in R' \setminus D, \end{cases} \tag{10}$$

then

$$\iint_{R'} F'(x,y)\,dA = \iint_R F(x,y)\,dA. \tag{11}$$

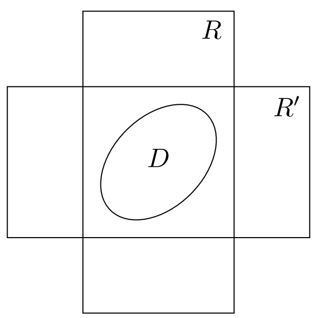


Figure 3: We can integrate using either of the rectangles R and R' enclosing the domain D if we extend the function f appropriately.

The properties of (double) integrals over rectangles (Propositions 1, 3 and 4) immediately carry over to general domains. We also get additivity if we combine non-overlapping domains.

Proposition 8 (Additivity of Double Integrals over Disjoint Domains)**.** *Let D_1 and D_2 be closed (and bounded) domains, let $f(x,y)$ be function defined on both D_1 and D_2*

If $\operatorname{int} D_1 \cap \operatorname{int} D_2 = \varnothing$, *then*

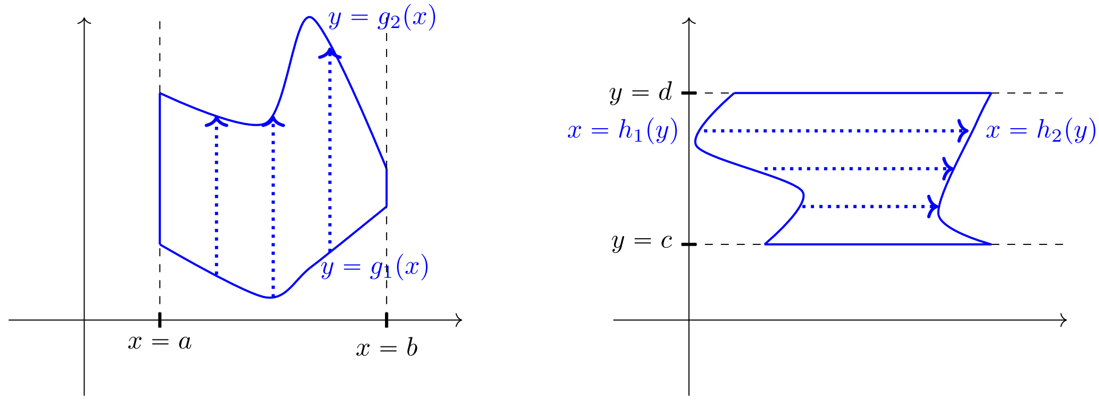
$$\iint_{D_1 \cup D_2} f\,dA = \iint_{D_1} f\,dA + \iint_{D_2} f\,dA. \tag{12}$$

Proof. Omitted. (Clear from the properties of Riemann sums.) □

0.1.1 Special Domains

There are two types of special closed (and bounded) domains that we want to consider:

- (Type I) $D = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\}$, where g_1 and g_2 are *continuous* functions on $[a, b]$ with $g_1(x) \leqslant g_2(x)$ for $x \in [a, b]$,
- (Type II) $D = \{(x, y) \in \mathbb{R}^2 \mid h_1(y) \leqslant x \leqslant h_2(y), c \leqslant y \leqslant d\}$, where h_1 and h_2 are *continuous* functions on $[c, d]$ with $h_1(x) \leqslant h_2(x)$ for $y \in [c, d]$.



A Type I Domain

A Type II Domain

Figure 4: Special domains in $\mathbb{R}^2$.

Remark 9. The terms Type I and Type II domains are not widely used in general but will be adopted in this course.

For these types of domains, we have

Theorem 10 (Fubini's Theorem (Stronger Version))**.** *Let $f(x, y)$ be a* continuous *function on a closed (and bounded) domain D.*

1. *If D is of Type I as above, then*

$$\iint\limits_D f(x, y)\, dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right] dx, \tag{13}$$

2. *If D is of Type II as above, then*

$$\iint\limits_D f(x, y)\, dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x, y)\, dx \right] dy. \tag{14}$$

Proof. We prove the Type I case. First, we choose the rectangle $R = [a, b] \times [c, d]$, where $c = \min\limits_{[a,b]} g_1(x)$ and $d = \max\limits_{[a,b]} g_2(x)$. This rectangle contains the original domain D.

By Definition 6, we have

$$\iint\limits_D f(x, y)\, dA = \iint\limits_R F(x, y)\, dA. \tag{15}$$

Since f is continuous on D, we see that the extension

$$F(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \in D, \\ 0, & \text{if } (x, y) \in R \backslash D, \end{cases} \tag{16}$$

is continuous on all of R except possibly on the boundary of D. By Proposition 3, F [**Exercise** : in fact also $|F|$] is integrable over R. We can now Fubini's Theorem to F on the rectangle $R = [a, b] \times [c, d]$. (Recall

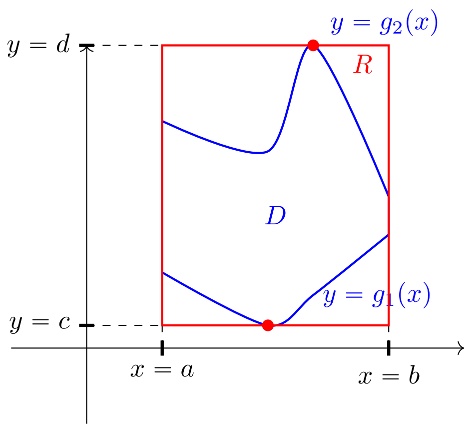


Figure 5: Choosing a rectangle R that encloses the domain D.

from Remark **??** that it also holds for absolutely integrable functions.) As such, we have

$$\iint_R F(x,y)\,dA = \int_a^b \left[\int_c^d F(x,y)\,dy \right] dx. \tag{17}$$

By definition, we have that $F(x,y) = 0$ for $y < g_1(x)$ and for $y > g_2(x)$. Also $F(x,y) = f(x,y)$ for $g_1(x) \leqslant y \leqslant g_2(x)$. Making the appropriate replacements, tt follows that

$$\int_a^b \left[\int_c^d F(x,y)\,dy \right] dx = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx. \tag{18}$$

Combining the previous few equalities yields the result.

The Type II domain case can be proven similarly. □

Example 11. Integrate the function $f(x,y) = 4y + 2$ over the region bounded by the curves $y = x^2$ and $y = 2x$.

Solution. We want to write the domain of integration as one of our special domains. We can do so by first noting where the curves intersect.

Setting them equal to each other, we get the equation

$$x^2 = 2x \tag{19}$$

which has solutions $x = 0$ and $x = 2$. These correspond to the points $(0,0)$ and $(2,4)$ respectively and the domain D can be written as

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant x \leqslant 2, x^2 \leqslant y \leqslant 2x\}. \tag{20}$$

Keep in mind which of these functions becomes the lower bound and which becomes the upper bound.

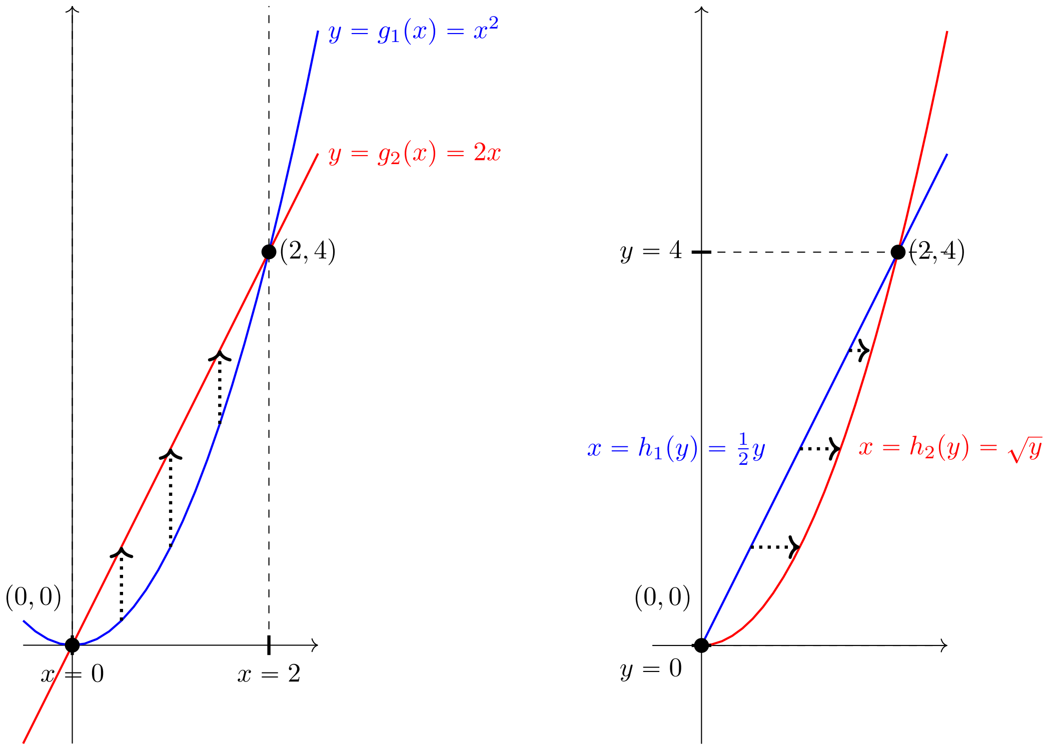


Figure 6: Writing the domain of integration as a Type I and a Type II domain.

Integrating then gives

$$\begin{aligned}\iint_D f(x,y)\,dA &= \int_0^2 \left[\int_{x^2}^{2x} 4y^2 + 2\,dy\right] dx \\ &= \int_0^2 \left[2y^2 + 2y\Big|_{y=x^2}^{y=2x}\right] dx \\ &= \int_0^2 \Big(-2x^4 + 6x^2 + 4x\Big)\,dx \\ &= \ldots = \frac{56}{5}. \qquad [\textbf{Exercise}: \text{Check this}] \qquad (21)\end{aligned}$$

Alternatively, we can write our domain as a Type II domain. This will require inverting the functions $y = x^2$ and $y = 2x$ to get expressions for x in terms of y.

Integrating this way, we get

$$\begin{aligned}\iint_D f(x,y)\,dA &= \int_0^4 \left[\int_{\frac{1}{2}y}^{\sqrt{y}} (4y+2)\,dx\right] dy \\ &= \ldots = \int_0^4 (4y+2)\Big(\sqrt{y} - \frac{1}{2}y\Big)\,dy \\ &= \ldots = \frac{56}{5}. \qquad [\textbf{Exercise}: \text{Check this}] \qquad (22)\end{aligned}$$

□

Example 12. Evaluate $\int_0^1 \left[\int_y^1 \frac{\sin x}{x}\,dx\right] dy$.

Solution. We can regard this as a double integral of the function $f(x,y) = \frac{\sin x}{x}$ over the domain

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \leqslant y \leqslant 1, y \leqslant x \leqslant 1\}. \tag{23}$$

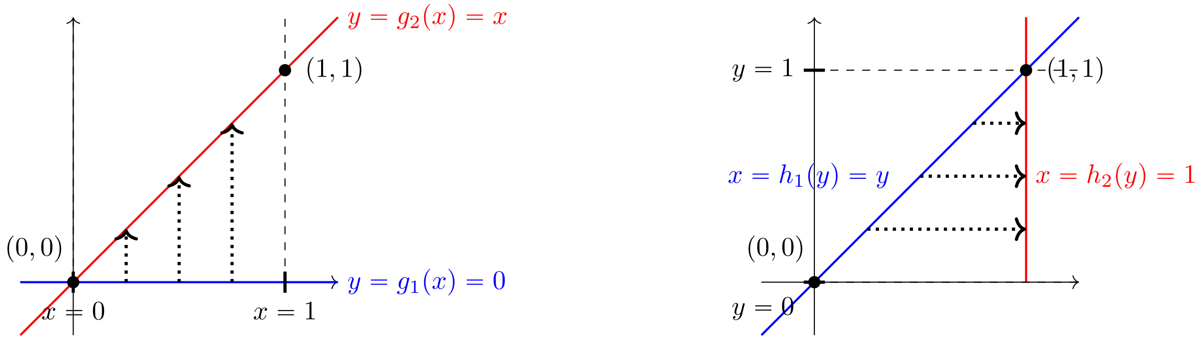


Figure 7: The domain of the double integral as a Type I and Type II domain.

By Fubini's Theorem, we can swap the order of integration (moving from a Type II domain to a Type I domain) and get

$$\int_0^1 \left[\int_y^1 \frac{\sin x}{x}\, dx \right] dy = \iint_D \frac{\sin x}{x}\, dA = \int_0^1 \left[\int_0^x \frac{\sin x}{x}\, dy \right] dx. \tag{24}$$

Continuing, we get

$$\int_0^1 \left[\int_0^x \frac{\sin x}{x}\, dy \right] dx = \int_0^1 \sin x\, dx = 1 - \cos 1. \tag{25}$$

□

Remark 13. Recall that the function $f(x,y) = \frac{\sin x}{x}$ is undefined when $x = 0$. [**Exercise** : Why does Fubini's Theorem apply here?] (Hint: Remember that $\lim_{x \to 0} \frac{\sin x}{x} = 1$.)

Example 14. Find $\iint_D x\, dA$, where D is the region in the right half-plane bounded by the lines $y = 0$, $x + y = 0$, and the unit circle.

Solution. We first get an idea of what this domain looks like.

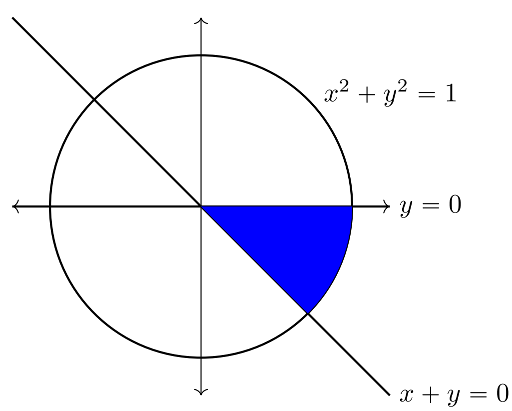


Figure 8: The domain of integration D.

In order to recognize this as a special domain, we will have to use the right part of the circle relation to get proper expressions of x in terms of y (or vice versa).

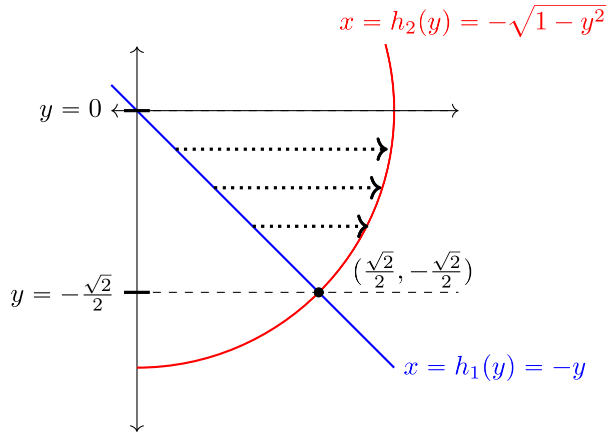


Figure 9: Writing D as a Type II domain.

By Fubini's Theorem, the integral is

$$\iint_D x\,dA = \int_{-\frac{\sqrt{2}}{2}}^{0} \left[\int_{-y}^{\sqrt{1-y^2}} x\,dx \right] dy = \int_{-\frac{\sqrt{2}}{2}}^{0} \left(\frac{1}{2} - y^2 \right) dy = \ldots = \frac{1}{3\sqrt{2}} \qquad [\textbf{Exercise}: \text{Check this.}] \tag{26}$$

As usual, we can do this computation using Type I domains. Here, we have to do something slightly tricky to get an expression for the lower curve by splitting it into two parts (using the line $x = \frac{\sqrt{2}}{2}$).

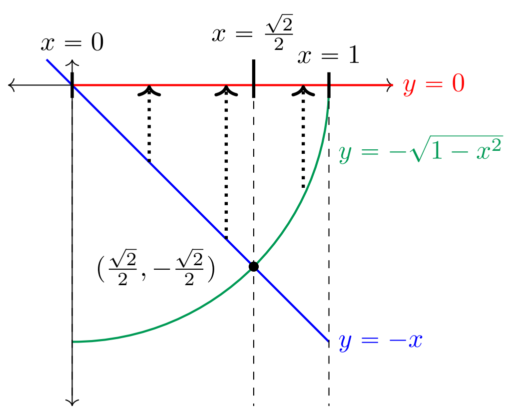


Figure 10: Writing D as a combination of two Type I domains.

This gives the computation

$$\begin{aligned} \iint_D x\,dA &= \int_0^{\frac{\sqrt{2}}{2}} \left[\int_{-x}^{0} x\,dy \right] dx + \int_{\frac{\sqrt{2}}{2}}^{1} \left[\int_{-\sqrt{1-x^2}}^{0} x\,dy \right] dx \\ &= \int_0^{\frac{\sqrt{2}}{2}} x^2\,dx + \int_{\frac{\sqrt{2}}{2}}^{1} x \cdot \sqrt{1-x^2}\,dx = \ldots = \frac{1}{3\sqrt{2}} \qquad [\textbf{Exercise}: \text{Check this.}] \end{aligned} \tag{27}$$

□

0.2 Applications of Double Integrals

We can interpret the (double) integral as several physical quantities.

(Signed) Volume

Definition 15 ((Signed) Volume of a Solid)**.** Let f be an integrable function over a "good" domain D in $\mathbb{R}^2$. The volume bounded by the surface $z = f(x, y)$ over the domain D is

$$\text{Volume} = \iint_D f(x, y)\, dA. \tag{28}$$

Here, any part of the surface where $f(x, y) > 0$ contributes positively to the volume, whereas any part when $f(x, y) < 0$ contributes negatively.

Area

Definition 16 (Area of a Domain)**.** Let D be a "good" domain in $\mathbb{R}^2$. The *area* of D is given by

$$\text{Area}(D) = \iint_R 1\, dA. \tag{29}$$

This can be seen as a consequence of the (signed) volume formula by setting the function $f(x, y) \equiv 1$.

Using this definition, Fubini's Theorem implies the well-known formula

$$\text{Area}(D) = \int_a^b \Big(f(x) - g(x)\Big)\, dx \tag{30}$$

when D is the region bounded by the curves $y = f(x)$ and $y = g(x)$ where $g(x) \leqslant f(x)$ on $[a, b]$, $f(a) = g(a)$ and $f(b) = g(b)$. [**Exercise** : Show this.]

Example 17. Find the area of the region bounded by $y = x^2$ and $y = x + 2$.

Solution. We first need to find the points of intersection. Setting the expressions equal to each other gives the equation

$$x^2 = x + 2 \tag{31}$$

which has solutions $x = -1$ and $x = 2$.

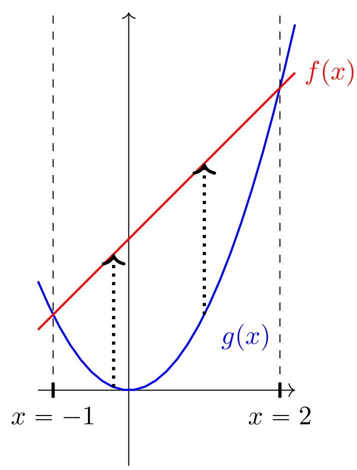


Figure 11: The domain R.

Applying the formula gives

$$\text{Area}(R) = \int_{-1}^{2} \Big(x + 2 - x^2\Big)\, dx = \ldots = \frac{9}{2}. \tag{32}$$

[**Exercise** : Check this.] □

Averages

Definition 18 (Average Value of a Function over a Domain)**.** Let f be an integrable function over a "good" domain D in $\mathbb{R}^2$. The *average value of f over D* is given by

$$\frac{1}{\text{Area}(D)} \iint_D f(x,y)\, dA. \tag{33}$$

We can think of this formula as "adding up" f over all of D, then dividing it evenly over the region.

Example 19. Let $f(x,y) = x\cos(xy)$ and $D = [0,\pi] \times [0,1]$. Find the average value of f over D.

Solution. Using the formula, we have

$$\begin{aligned}
\text{Avg Value of } f \text{ over } D &= \frac{1}{\text{Area}(D)} \iint_D f(x,y)\, dA \\
&= \frac{1}{\pi} \int_0^\pi \left[\int_0^1 x\cos(xy)\, dy \right] dx \\
&= \frac{1}{\pi} \int_0^\pi \sin x\, dx = \ldots = \frac{2}{\pi}. \qquad [\textbf{Exercise} : \text{Check this.}]
\end{aligned} \tag{34}$$

□

____________________(End of Lecture 3 – Sep 15)