MATH2020A Lecture 2 Notes

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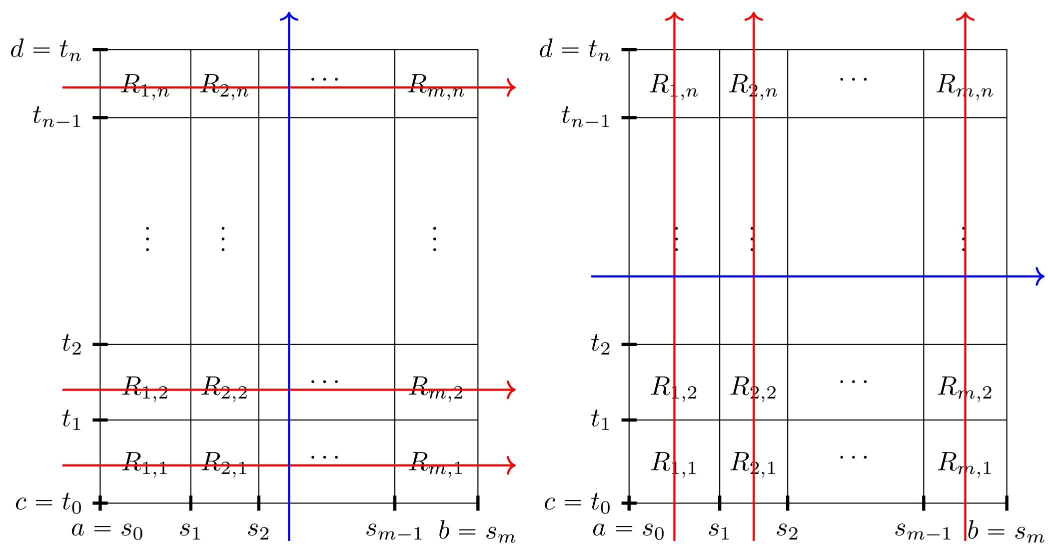
Last week, we defined double integrals over rectangles as limits of 2-dimensional Riemann sums. We did an example this way, which resulted in a long and tedious computation. The following result allows us to simplify these calculations.

Theorem 1 (Fubini's Theorem (First Form))**.** *If* $f(x, y)$ *is a* continuous *function on the rectangle* $R = [a, b] \times [c, d]$*, then*

$$\iint\limits_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx. \tag{1}$$

Idea of Proof. When summing over the sub-rectangles in the Riemann sum, we can either sum horizontally (over x) first, then vertically (over y), or sum vertically (over y) first, then horizontally (over x).



$\int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$

Sum horizontally (over x) first,
then vertically (over y).

$\int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$

Sum vertically (over y) first,
then horizontally (over x).

Figure 1: Using different methods to sum up over the sub-rectangles in a Riemann sum.

□

In essence, this result allows us to turn double integrals into iterated (single) integrals. We can now use all our usual integration tricks to simplify these. Keep in mind that when performing these iterated integrals, we treat the variable that we are not integrating as a constant.

Remark 2. When writing these iterated integrals, we sometimes drop the brackets around the inside integral. The order of the terms dx and dy tell us which way we are integrating first (from inside out).

Remark 3 (Absolutely Integrable Functions)**.** We note here that Fubini's Theorem also holds if the function $f(x, y)$ is *absolutely integrable* over R. That is if the function $f(x, y)$ on R is such that

$$\iint\limits_R |f(x,y)| \, dA < \infty, \tag{2}$$

then

$$\iint\limits_R f(x,y)\, dA = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx. \tag{3}$$

The notion of being *absolutely integrable* over R is a weaker condition than being *continuous* over R.

Example 4. Let $R = [0, 2] \times [0, 1]$ and $f(x, y) = xy^2$. Using Fubini's theorem, find $\iint_R f(x, y)\, dx\, dy$.

Solution. By Fubini's theorem, we have

$$\begin{aligned} \iint\limits_R f(x,y)\, dx\, dy &= \int_0^2 \left[\int_0^1 xy^2\, dy \right] dx = \int_0^2 \left[x \cdot \frac{y^3}{3} \bigg|_{y=0}^{y=1} \right] dx \\ &= \int_0^2 \frac{x}{3}\, dx = \frac{x^2}{6} \bigg|_{x=0}^{x=2} = \frac{2}{3} \end{aligned}$$

□

We see that this ends up being much simpler than the previous example.

Sometimes, the order in which we perform the integrals matters for our calculations.

Example 5. Find $\iint_{[0,1]\times[0,\pi]} x \sin(xy)\, dA$.

Solution. If we integrate over x first, we get

$$\begin{aligned} \iint\limits_{[0,1]\times[0,\pi]} x\sin(xy)\, dA &= \int_0^\pi \left[\int_0^1 x \sin(xy)\, dx \right] dy \\ &= \int_0^\pi \left[-\frac{x\cos(xy)}{y} \bigg|_{x=0}^{x=1} - \int_0^1 -\frac{\cos(xy)}{y}\, dx \right] dy \qquad \text{[Integration by Parts]} \\ &= \int_0^\pi \left[-\frac{x\cos(xy)}{y} + \frac{\sin(xy)}{y^2} \bigg|_{x=0}^{x=1} \right] dy \\ &= \int_0^\pi \left(-\frac{\cos(y)}{y} + \frac{\sin(y)}{y^2} \right) dy \\ &= \dots . \end{aligned} \tag{4}$$

We run into some trouble here since these integrals are tricky.

If we instead integrate over y first, we get

$$\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^1 \left[\int_0^\pi x\sin(xy)\,dy\right]dx$$
$$= \int_0^1 \left[-\cos(xy)\Big|_{y=0}^{y=\pi}\right]dx$$
$$= \int_0^1 \Big(-\cos(\pi x)+1\Big)\,dx$$
$$= \Big(-\frac{1}{\pi}\sin(\pi x)+x\Big)\Big|_{x=0}^{x=1} = 1,$$

which was much easier to compute. □

Not all functions are integrable over a (closed) rectangle.

Remark 6. To show a function f is *integrable*, we need to show that the Riemann sums

$$S(f,P) = \sum_{k=1}^{m}\sum_{l=1}^{n} f(x_{k,l},y_{k,l})\cdot\Delta A_{k,l} \tag{5}$$

all approach the *same* limit as $\|P\| \to 0$ *regardless* of which partitions P and points $(x_{k,l},y_{k,l}) \in R_{k,l}$ we choose.

Conversely, to prove that a function f is *not integrable*, we need to find

1. either partitions P and points $(x_{k,l},y_{k,l}) \in R_{k,l}$ such that
$$\lim_{\|P\|\to 0} S(f,P) \tag{6}$$
does not exist,
2. or partitions P and P' with different points $(x_{k,l},y_{k,l}) \in R_{k,l}$ and $(x'_{k,l},y'_{k,l}) \in R'_{k,l}$ such that
$$\lim_{\|P\|\to 0} S(f,P,x_{k,l},y_{k,l}) = a \neq b = \lim_{\|P'\|\to 0} S(f,P',x_{k,l},y_{k,l}). \tag{7}$$
(We may use the same partitions for P and P' as long as the points chosen are different.)

Example 7 (A Non-Integrable Function)**.** Let $R = [0,1]\times[0,1]$ and

$$f(x,y) = \begin{cases} 0, & \text{if both } x \text{ and } y \text{ are rational,} \\ 1, & \text{otherwise.} \end{cases} \tag{8}$$

Show that f is *not integrable* over R (using point 2 in Remark 6).

Solution. Note that for any partition P of R, we can find points $(x_{k,l},y_{k,l}) \in R_{k,l}$ in each sub-rectangle such that both $x_{k,l}$ and $y_{k,l}$ are rational. [**Exercise** : Why is this the case?]

The corresponding Riemann sums are

$$S(f,P,x_{k,l},y_{k,l}) = \sum_{k=1}^{m}\sum_{l=1}^{n} f(x_{k,l},y_{k,l})\cdot\Delta A_{k,l} = \sum_{k=1}^{m}\sum_{l=1}^{n} 0\cdot\Delta A_{k,l} = 0 \to 0 \text{ as } \|P\|\to 0 \tag{9}$$

On the other hand, for any partition P of R we can also find points $(x'_{k,l},y'_{k,l}) \in R_{k,l}$ in each sub-rectangle such that at least one of $x'_{k,l}$ or $y'_{k,l}$ is irrational. [**Exercise** : Why is this the case?]

The corresponding Riemann sums are then

$$S(f,P,x'_{k,l},y'_{k,l}) = \sum_{k=1}^{m}\sum_{l=1}^{n} f(x'_{k,l},y'_{k,l})\cdot\Delta A_{k,l} = \sum_{k=1}^{m}\sum_{l=1}^{n} 1\cdot\Delta A_{k,l} = 1 \to 1 \text{ as } \|P\|\to 0. \tag{10}$$

Since these limits do not match up, we conclude that f is *not integrable.* □

Example 8 (A Non-Integrable Function II)**.** Let $R = [0,1] \times [0,1]$ and

$$f(x,y) = \begin{cases} \frac{1}{xy}, & \text{if } x \neq 0 \text{ and } y \neq 0, \\ 0, & \text{if } x = 0 \text{ or } y = 0. \end{cases} \tag{11}$$

Show that f is *not integrable* over R (using point 1 in Remark 6).

Solution. In any partition P of R, we must have a sub-rectangle $R_{1,1} = [0, s_1] \times [0, t_1]$. Choose the point $(x_{1,1}, y_{1,1}) = (s_1^2, t_1^2) \in R_{1,1}$ (recall that $s_1, t_1 \leqslant 1$ and so $s_1^2 \leqslant s_1 \leqslant 1$ and $t_1^2 \leqslant t_1 \leqslant 1$).

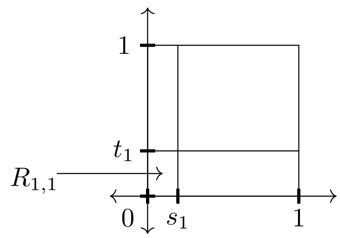


Figure 2: A partition P of the rectangle $R = [0,1] \times [0,1]$ will have a sub-rectangle $R_{1,1} = [0, s_1] \times [0, t_1]$ in the lower-left corner.

The Riemann sum is then

$$\begin{aligned} S(f,P) &= \sum_{k=1}^{m} \sum_{l=1}^{n} f(x_{k,l}, y_{k,l}) \cdot \Delta A_{k,l} \\ &= f(x_{1,1}, y_{1,1}) \cdot \Delta A_{1,1} + \underbrace{\sum_{\substack{1 \leqslant k \leqslant m \\ 1 \leqslant l \leqslant n \\ (k,l) \neq (1,1)}} f(x_{k,l}, y_{k,l}) \cdot \Delta A_{k,l}}_{\geqslant 0} \\ &\geqslant \frac{1}{s_1^2 t_1^2} \cdot s_1 t_1 = \frac{1}{s_1 t_1}. \end{aligned} \tag{12}$$

Since $0 < s_1, t_1 \leqslant \|P\|$, we have that $s_1, t_1 \to 0$ as $\|P\| \to 0$. Hence

$$S(f,P) \geqslant \frac{1}{s_1 t_1} \to \infty \text{ as } \|P\| \to 0. \tag{13}$$

The limit of Riemann sums does not exist and so f is *not integrable.* ☐

(End of Lecture 2 – Sep 11)