MATH2020A Lecture 20 Notes

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Recall the statement of the Divergence Theorem:

Theorem 1 (Divergence Theorem). *Let $\vec{F}$ be a C^1 vector field on an open domain $\Omega \subseteq \mathbb{R}^3$. Let S be a piecewise smooth oriented closed (having no boundary) surface enclosing a (solid) region $D \subseteq \Omega$. Let $\hat{\mathbf{n}}$ be the outward-pointing unit normal vector field on S. Then*

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}} \, d\sigma = \iiint_D \operatorname{div} \vec{F} \, dV = \iiint_D \vec{\nabla} \cdot \vec{F} \, dV. \tag{1}$$

We will prove this result for a special case.

Proof of the Divergence Theorem. We will only prove this in the case of a special domain D which is simultaneously of Type I, II, and III.

Recall that this means that there exist regions R_1, R_2, and R_3 in $\mathbb{R}^2$ and functions f_1, f_2, g_1, g_2, h_1, and h_2 such that

$$
\begin{aligned}
D &= \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in R_1, \quad f_1(x,y) \le z \le f_2(x,y)\} &&\text{[Type I]} \\
&= \{(x,y,z) \in \mathbb{R}^3 \mid (x,z) \in R_2 \quad g_1(x,z) \le y \le g_2(x,z)\} &&\text{[Type II]} \\
&= \{(x,y,z) \in \mathbb{R}^3 \mid (y,z) \in R_3 \quad h_1(y,z) \le x \le h_2(y,z)\}. &&\text{[Type III]}
\end{aligned}
\tag{2}
$$

Like we did for the proof of Green's Theorem, given

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}}, \tag{3}$$

we will prove three equalities that combine together to give the result.

In particular, we want to show that

$$
\begin{cases}
\displaystyle \iint_S M\,\hat{\mathbf{i}} \cdot \hat{\mathbf{n}} \, d\sigma &= \displaystyle \iiint_D \frac{\partial M}{\partial x} \, dV, & \text{[by Type III]} \\[2ex]
\displaystyle \iint_S N\,\hat{\mathbf{j}} \cdot \hat{\mathbf{n}} \, d\sigma &= \displaystyle \iiint_D \frac{\partial N}{\partial y} \, dV, & \text{[by Type II]} \\[2ex]
\displaystyle \iint_S L\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \, d\sigma &= \displaystyle \iiint_D \frac{\partial L}{\partial z} \, dV. & \text{[by Type I]}
\end{cases}
\tag{4}
$$

The proofs of all three are similar so we will only show the last one:

$$\iint_S L\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \, d\sigma = \iiint_D \frac{\partial L}{\partial z} \, dV. \tag{5}$$

By Fubini's Theorem, since D is Type I, the RHS is
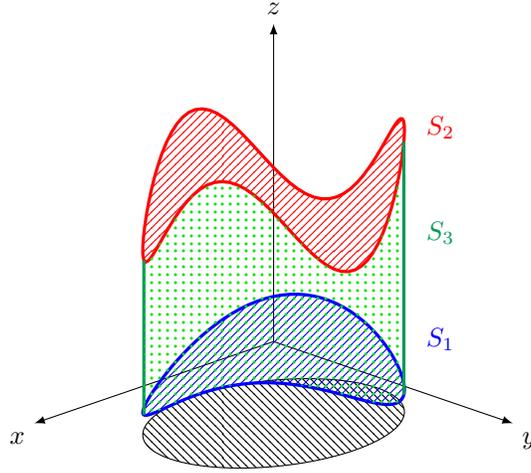
$$
\begin{aligned}
\text{RHS} &= \iiint_D \frac{\partial L}{\partial z} \, dV \\
&= \iint_{R_1} \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial L}{\partial z} \, dz \right] dx \, dy \\
&= \iint_{R_1} \left[L\big(x, y, f_2(x,y)\big) - L\big(x, y, f_1(x,y)\big) \right] dx \, dy.
\end{aligned}
\tag{6}
$$

For the LHS, we note that by the definition of a Type I domain, the boundary $S = \partial D$ can be written as
$$S = S_1 \cup S_2 \cup S_3 \tag{7}$$
where

- S_1 is the graph of f_1 over R_1,
- S_2 is the graph of f_2 over R_1,
- S_3 is a "vertical" surface between S_1 and S_2.

Figure 1: The boundary of a Type I 3D domain can be decomposed into three parts.

Hence
$$\begin{aligned}
\text{LHS} &= \iint_S L\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,d\sigma \\
&= \iint_{S_1} L\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,d\sigma + \iint_{S_2} L\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,d\sigma + \iint_{S_3} L\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,d\sigma.
\end{aligned} \tag{8}$$

The integral over S_3 vanishes since S_3 is a "vertical" surface, and so the unit normal $\hat{\mathbf{n}}$ there has no $\hat{\mathbf{k}}$ component.

For the upper surface S_2, the outward-pointing unit normal $\hat{\mathbf{n}}$ points upwards (that is, has non-negative $\hat{\mathbf{k}}$-component). Note that we have the natural parameterization
$$(x, y) \mapsto \vec{r}(x, y) = \big(x, y, f_2(x, y)\big) \tag{9}$$
where
$$\begin{cases} \vec{r}_x &= \hat{\mathbf{i}} + \dfrac{\partial f_2}{\partial x}\,\hat{\mathbf{k}}, \\[2mm] \vec{r}_y &= \hat{\mathbf{j}} + \dfrac{\partial f_2}{\partial y}\,\hat{\mathbf{k}} \end{cases} \tag{10}$$
and
$$\vec{r}_x \times \vec{r}_y = -\frac{\partial f_2}{\partial x}\,\hat{\mathbf{i}} - \frac{\partial f_2}{\partial y}\,\hat{\mathbf{j}} + \hat{\mathbf{k}}. \tag{11}$$

With this, the outward unit normal $\hat{\mathbf{n}}$ is given by
$$\hat{\mathbf{n}} = \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|} \implies \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} = \frac{1}{\|\vec{r}_x \times \vec{r}_y\|}. \tag{12}$$

Therefore,

$$\iint_{S_2} L\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,d\sigma = \iint_{R_1} L\big(x, y, f_2(x,y)\big)\frac{1}{\|\vec{r}_x \times \vec{r}_y\|}\,\|\vec{r}_x \times \vec{r}_y\|\,dx\,dy$$
$$= \iint_{R_1} L\big(x, y, f_2(x,y)\big)\,dx\,dy. \tag{13}$$

For the lower surface S_1, we have a similar computation, except that the unit normal points downwards.

Using the natural parameterization

$$(x, y) \mapsto \vec{r}(x, y) = \big(x, y, f_1(x,y)\big), \tag{14}$$

we have

$$\hat{\mathbf{n}} = -\frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|} \implies \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} - \frac{1}{\|\vec{r}_x \times \vec{r}_y\|}. \tag{15}$$

As such,

$$\iint_{S_1} L\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,d\sigma = -\iint_{R_1} L\big(x, y, f_1(x,y)\big)\,dx\,dy. \tag{16}$$

Putting everything together, we see that

$$\iint_{S} L\,\hat{\mathbf{k}} \cdot \hat{\mathbf{n}}\,d\sigma = \iint_{R_1} L\big(x, y, f_2(x,y)\big)\,dx\,dy - \iint_{R_1} L\big(x, y, f_1(x,y)\big)\,dx\,dy$$
$$= \iint_{R_1} \Big[L\big(x, y, f_2(x,y)\big) - L\big(x, y, f_1(x,y)\big) \Big]\,dx\,dy$$
$$= \iiint_{D} \frac{\partial L}{\partial z}\,dV, \tag{17}$$

which proves the result. $\qquad\square$

Remark 2. Similarly to Green's Theorem, the Divergence Theorem also holds for solid regions with finitely many "holes" inside so long as the surfaces are oriented appropriately.

To prove this general version, we make "cuts" to the solid to "expose" the holes to the boundary analogously to the method used for Green's Theorem. The newly exposed surfaces are double-counted, but with opposite orientations, which cancel out in the end (see figure on next page).

Remark 3. The notion of *flux density* from before also applies in this case via the Divergence Theorem.

(End of Lecture 20 – Nov 20)
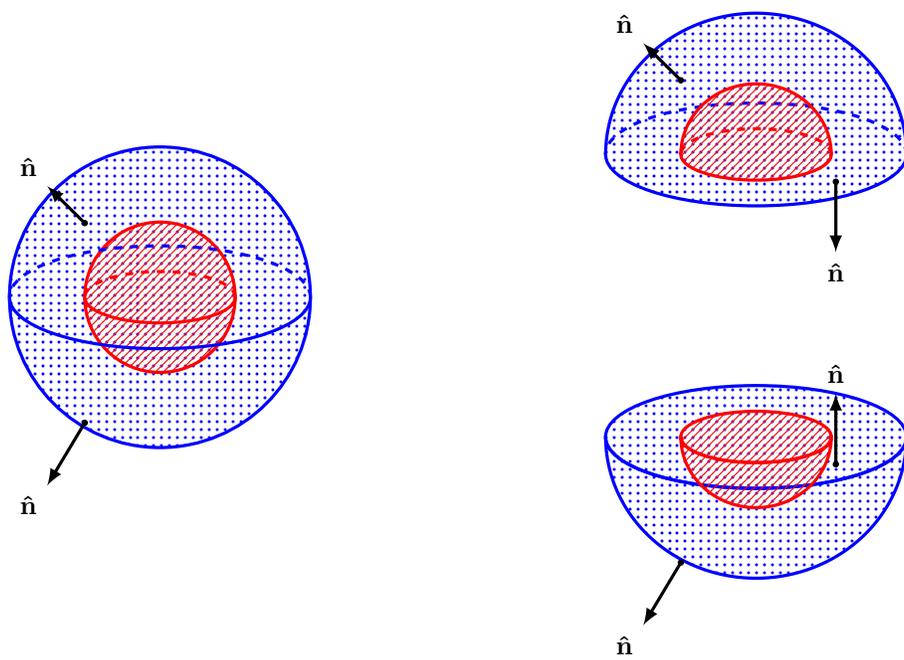
Figure 2: Cutting a solid in the proof of the general case of the Divergence Theorem.