MATH2020A Lecture 1 Notes

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These lecture notes are based on Prof. Tom Wan's notes from the 2024/25 offering of the course. The course is split into two sections: the former covering integration in 2 and 3 dimensions and the latter covering vector analysis.

1 Integration in 2 and 3 Dimensions

1.1 Double Integrals over Rectangles

Recall that in 1 dimension, an *integral* is defined as a *limit* of *Riemann sums*. We write

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k)\cdot \Delta x_k, \tag{1}$$

where

- f is a function on the (closed) interval $[a, b]$,
- P is a *partition* $a = t_0 < t_1 < t_2 < \ldots < t_n = b$ of $[a, b]$ with n parts,
- $x_k \in [t_{k-1}, t_k]$,
- $\Delta x_k = t_k - t_{k-1}$,
- $\|P\| = \max_k |\Delta x_k|$.

In the *limit*, we are really considering a whole family of partitions that get finer and finer and seeing if these Riemann sums approach a specific number.

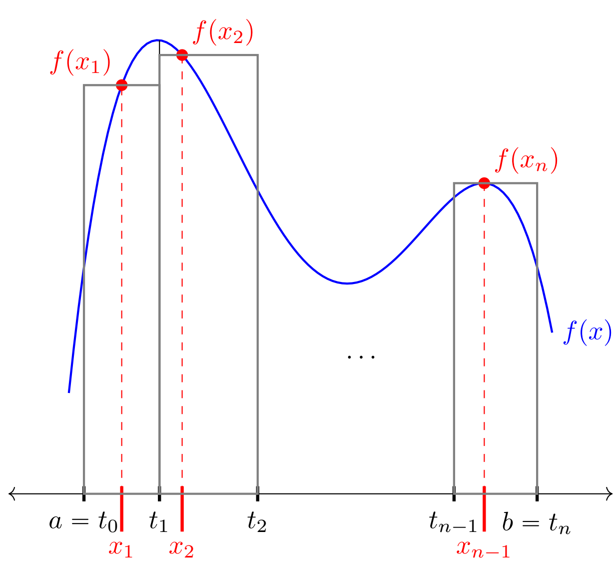

Figure 1: A Riemann sum of the function $f(x)$ on the interval $[a, b]$ using partition $P\colon a = t_0 < \ldots < t_n = b$.

What is happening here is that we are summing up the areas of small rectangles (in gray) with "height" $f(x_k)$ and "base" $t_k - t_{k-1}$. As the partition gets finer and finer, we get better and better approximations of the (signed) area.

Remark 1. In practice, for computations we usually use *uniform partitions* of our intervals. These have the intermediate points evenly spaced out. For an interval $[a, b]$ the uniform partition P with n parts is given by

$$a = t_0 < t_1 = a + \frac{b-a}{n} < t_2 = a + 2 \cdot \left(\frac{b-a}{n}\right) < \ldots < t_k = a + k \cdot \left(\frac{b-a}{n}\right) < \ldots < t_n = b. \tag{2}$$

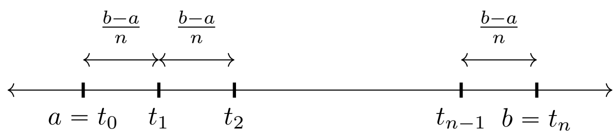

Figure 2: A uniform partition of the interval $[a, b]$ with n parts.

In this case, $\|P\| = \max_k |\Delta x_k| = \frac{b-a}{n} \to 0$ as $n \to \infty$. Our integral computation then becomes

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k) \cdot \Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k) \cdot \left(\frac{b-a}{n}\right). \tag{3}$$

Example 2. Find $\int_0^1 x^3\, dx$ (*i.e.* find the integral of $f(x) = x^3$ on the interval $[0, 1]$).

Solution. Let us use uniform partitions on the interval $[0, 1]$. By doing this, we have $t_k = \frac{k}{n}$ for $0 \leqslant k \leqslant n$. We are still free to choose the points $x_k \in [t_{k-1}, t_k]$, we will use two different methods for this:

- Method 1 (Left-Endpoint Method): On each interval $[t_{k-1}, t_k] = [\frac{k-1}{n}, \frac{k}{n}]$, we set $x_k = t_{k-1} = \frac{k-1}{n}$ as the left-endpoint. Plugging into our formula, we have the Riemann sums

$$S_n = \sum_{k=1}^{n} f(x_k) \cdot \Delta x_k = \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^3 \cdot \frac{1}{n} = \underbrace{\ldots}_{\textbf{Exercise}} = \frac{1}{4}\left(1 - \frac{1}{n}\right)^2 \to \frac{1}{4} \text{ as } n \to \infty. \tag{4}$$

  As such, we get $\int_0^1 x^3 dx = \frac{1}{4}$.

- Method 2 (Right-Endpoint Method): On each interval $[t_{k-1}, t_k] = [\frac{k-1}{n}, \frac{k}{n}]$, we set $x_k = t_k = \frac{k}{n}$ as the right-endpoint. Plugging into our formula, we have the Riemann sums

$$S_n = \sum_{k=1}^{n} f(x_k) \cdot \Delta x_k = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \cdot \frac{1}{n} = \underbrace{\ldots}_{\textbf{Exercise}} = \frac{1}{4}\left(1 + \frac{1}{n}\right)^2 \to \frac{1}{4} \text{ as } n \to \infty. \tag{5}$$

  As such, we get $\int_0^1 x^3 dx = \frac{1}{4}$.

□

Remark 3. We are free to choose *any* $x_k \in [t_{k-1}, t_k]$ to still get the same result. In practice, we use whichever method ends up with the cleanest calculations in the end.

This concept of integration can be generalized to *any* dimension.

For 2 dimensions, we first consider a function $f(x, y)$ defined on a (closed and bounded) rectangle $R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$. Just like we did with the interval, we want to subdivide R into sub-rectangles by using partitions. We will need a partition

$$P_1\colon a = s_0 < s_1 < s_2 < \ldots s_m = b \text{ of } [a, b], \tag{6}$$

and another partition

$$P_2\colon c = t_0 < t_1 < t_2 < \ldots t_n = b \text{ of } [c, d]. \tag{7}$$

Note that these partitions do not need to have the same number of parts.

Denote $P = P_1 \times P_2$ and $\|P\| = \max(\|P_1\|, \|P_2\|)$. The partition P will divide $R = [a, b] \times [c, d]$ into mn sub-rectangles $R_{k,l}$, each with some area $\Delta A_{k,l}$.

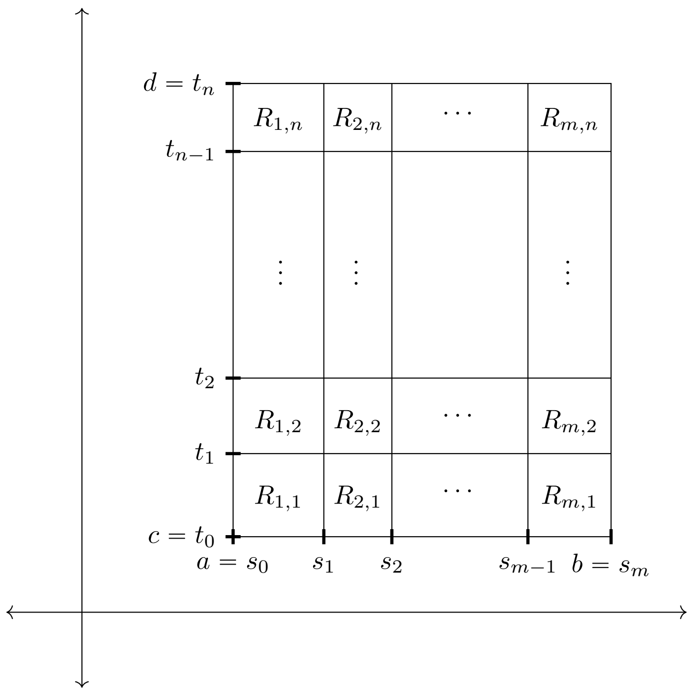

Figure 3: A partition P of the rectangle $R = [a, b] \times [c, d]$ subdivides R into smaller sub-rectangles $R_{k,l}$.

We can then choose a point $(x_{k,l}, y_{k,l})$ in each sub-rectangle $R_{k,l}$ and consider the Riemann sum

$$S(f, P, x_{k,l}, y_{k,l}) = \sum_{k=1}^{m} \sum_{l=1}^{n} f(x_{k,l}, y_{k,l}) \cdot \Delta A_{k,l}. \tag{8}$$

Note that the f and P are there to remind us that this sum depends on the function f, the partitions P that we chose, and also the points $(x_{k,l}, y_{k,l})$ that we chose (we sometimes drop some of this notation when some of the information is clear by context). Taking the *limit* of *Riemann sums* as partition size approaches zero will give us the integral.

Definition 4. A function f is said to be *integrable* over a rectangle R if

$$\lim_{\|P\| \to 0} S(f, P, x_{k,l}, y_{k,l}) = \lim_{\|P\| \to 0} \sum_{k=1}^{m} \sum_{l=1}^{n} f(x_{k,l}, y_{k,l}) \cdot \Delta A_{k,l} \tag{9}$$

exists and is *independent* of the choice of partitions P and $(x_{k,l}, y_{k,l}) \in R_{k,l}$.

In this case, the limit is called the *(double) integral of* f *over* R and is denoted by

$$\iint\limits_R f(x, y)\, dA \qquad \text{or} \qquad \iint\limits_R f(x, y)\, dx\, dy. \tag{10}$$

Remark 5. As was the case in 1 dimension, the double integral of a function f can be interpreted as the (signed) *volume* under the graph of f (over R).

In particular, when $f \equiv 1$, we get

$$\iint\limits_R 1 dA = \text{Area of } R. \tag{11}$$

Example 6. Let $R = [0, 2] \times [0, 1]$ and $f(x, y) = xy^2$. Using the Riemann sum definition, find $\iint_R f(x, y)\, dx\, dy$.

Solution. Let us use uniform partitions of the intervals $[0, 2]$ and $[0, 1]$ to make things easier.

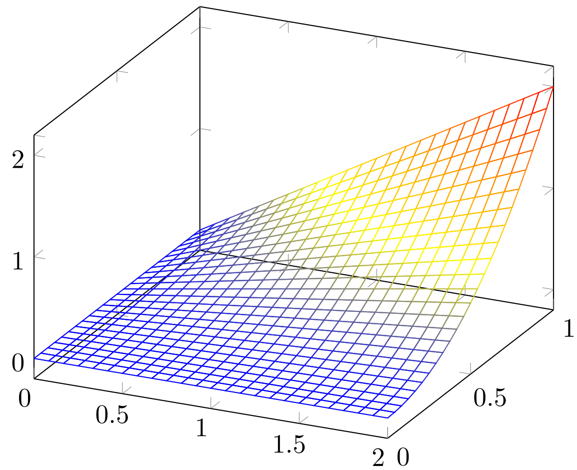

Figure 4: A plot of the function $f(x, y) = xy^2$.

We pick the partition

$$P_1 \colon 0 < \frac{2}{n} < \frac{4}{n} < \ldots < 2 \text{ of } [0, 2] \tag{12}$$

and the partition

$$P_2 \colon 0 < \frac{1}{n} < \frac{2}{n} < \ldots 1 \text{ of } [0, 1]. \tag{13}$$

(Here, the partitions we chose have the same number of parts.)

Our sub-rectangles are of the form $R_{k,l} = [\frac{2(k-1)}{n}, \frac{2k}{n}] \times [\frac{l-1}{n}, \frac{l}{n}]$. Each of these has area $\Delta A_{k,l} = \frac{2}{n} \cdot \frac{1}{n} = \frac{2}{n^2}$.

In each sub-rectangle $R_{k,l}$, we can pick the point $(x_{k,l}, y_{k,l}) = (\frac{2k}{n}, \frac{l}{n})$. This gives the Riemann sum

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \sum_{l=1}^{n} f(x_{k,l}, y_{k,l}) \cdot \Delta A_{k,l} \\ &= \sum_{k=1}^{n} \sum_{l=1}^{n} \left(\frac{2k}{n}\right) \cdot \left(\frac{l}{n}\right)^2 \cdot \left(\frac{2}{n^2}\right) \\ &= \frac{4}{n^5} \cdot \sum_{k=1}^{n} \left(\sum_{l=1}^{n} kl^2 \right) \\ &= \frac{4}{n^5} \cdot \sum_{k=1}^{n} \left(k \sum_{l=1}^{n} l^2 \right) \\ &= \frac{4}{n^5} \cdot \left(\sum_{k=1}^{n} k \right) \cdot \left(\sum_{l=1}^{n} l^2 \right) \\ &= \frac{4}{n^5} \cdot \frac{n(n+1)}{2} \cdot \frac{n(n+1)(2n+1)}{6} \to \frac{2}{3} \text{ as } n \to \infty. \end{aligned} \tag{14}$$

From this, we see that

$$\iint\limits_{[0,2]\times[0,1]} xy^2 dx\, dy = \frac{2}{3}. \tag{15}$$

□

The previous example required a very tedious calculation. The following theorem will help with this:

Theorem 7 (Fubini's Theorem (First Form)). *If* $f(x,y)$ *is a* continuous *function on the rectangle* $R = [a,b] \times [c,d]$*, then*

$$\iint\limits_R f(x,y)\,dA = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx. \tag{16}$$

———————————————————————————(End of Lecture 1 – Sep 4)