MATH2020A Lecture 19 Notes

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We start by taking another look at the consequences of Stokes' Theorem. In particular, how we can consider alternate surfaces to integrate over, as long as they have the same (oriented) boundary.

Example 1. Let
$$\vec{F} = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}}. \tag{1}$$
Let S be the surface obtained by intersecting the sphere of radius 3 centered at the origin and the half-space $x + y + z \geq 0$, that is
$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 9, x + y + z \geq 0\}, \tag{2}$$
with outward-pointing unit normal.

Find
$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma. \tag{3}$$

Solution. We first try to compute the integral explicitly.

One can check the radial vector field
$$x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \tag{4}$$
is normal at every point on a sphere centered at the origin.

Since S is part of the sphere of radius 3, we can normalize and get
$$\hat{\mathbf{n}} = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{3}.$$

A quick computation shows that
$$\operatorname{curl} \vec{F} = -2\,\hat{\mathbf{k}}. \tag{5}$$

This means that the flux integral becomes
$$\begin{aligned}\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma &= \iint_S (-2\,\hat{\mathbf{k}}) \cdot \left(\frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{3}\right) d\sigma \\ &= \iint_S -\frac{2z}{3}\, d\sigma. \end{aligned} \tag{6}$$

This integral is certainly doable from our techniques, but is quite tedious. An alternate method is to use Stokes' Theorem.

The boundary of this surface is
$$C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 9, x + y + z = 0\}. \tag{7}$$

By Stokes' Theorem, we see that
$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma = \oint_C \vec{F} \cdot d\vec{r}. \tag{8}$$

We also notice that C is the boundary of the disc
$$S' = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 9, x + y + z = 0\}. \tag{9}$$

This is a disc of radius 3 in $\mathbb{R}^3$ that lies in the plane $x + y + z = 0$.

Stokes' Theorem again tells us that

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{S'} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma \tag{10}$$

as long as we orient S' appropriately. One can check that taking the upward-pointing unit normal is consistent with the orientation on C induced from S.

Since S' lies in the plane, we can read off a normal vector in

$$\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}. \tag{11}$$

Normalizing this, gives a unit normal

$$\hat{\mathbf{n}} = \frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{3}} \tag{12}$$

on S'.

Thus have

$$\begin{aligned}\oint_C \vec{F} \cdot d\vec{r} &= \iint_{S'} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma \\ &= \iint_{S'} (-2\hat{\mathbf{k}}) \cdot \left(\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{3}} \right) d\sigma \\ &= -\frac{2}{\sqrt{3}} \cdot \iint_{S'} d\sigma \\ &= -\frac{2}{\sqrt{3}} \cdot \text{Area}(S') \\ &= -\frac{18\pi}{\sqrt{3}}. \end{aligned} \tag{13}$$

□

Proof of Stokes' Theorem. We will prove this in the special case where S is a graph given by the C^2 function $z = f(x, y)$ over a region R with upward-pointing unit normal.

Suppose C is the boundary of S and orient it accordingly. Also suppose C' is the boundary of the region $R \subseteq \mathbb{R}^2$ and orient it with respect to $\hat{\mathbf{k}}$.

We can parameterize S using

$$\vec{r}(x, y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + f(x, y)\,\hat{\mathbf{k}}, \quad (x, y) \in R. \tag{14}$$

As we have seen before, the partial derivatives give

$$\vec{r}_x = \hat{\mathbf{i}} + f_x\,\hat{\mathbf{k}}, \tag{15}$$

$$\vec{r}_y = \hat{\mathbf{j}} + f_y\,\hat{\mathbf{j}} \tag{16}$$

and taking the cross-product gives

$$\vec{r}_x \times \vec{r}_y = -f_x\,\hat{\mathbf{i}} - f_y\,\hat{\mathbf{j}} + \hat{\mathbf{k}}.$$

This vector indeed points upward since its $\hat{\mathbf{k}}$-component is positive.

From this, we see that

$$\hat{\mathbf{n}} = \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|} \tag{17}$$

is the upward unit normal of S and

$$d\sigma = \|\vec{r}_x \times \vec{r}_y\|\, dx\, dy = \|\vec{r}_x \times \vec{r}_y\|\, dA \tag{18}$$

is the area element of R.

Let

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}} \tag{19}$$

be a C^1-vector field. Then

$$\begin{aligned}
\iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma &= \iint_R (\vec{\nabla} \times \vec{F})\big(\vec{r}(x,y)\big) \cdot \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|} \|\vec{r}_x \times \vec{r}_y\|\, dA \\
&= \iint_R \Big((L_y - N_z)\,\hat{\mathbf{i}} + (M_z - L_x)\,\hat{\mathbf{j}} + (N_x - M_y)\,\hat{\mathbf{k}} \Big) \cdot \Big(- f_x\,\hat{\mathbf{i}} - f_y\,\hat{\mathbf{j}} + \hat{\mathbf{k}} \Big)\, dA \\
&= \iint_R \Big(- f_x(L_y - N_z) - f_y(M_z - L_x) + (N_x - M_y) \Big)\, dA. \qquad (20)
\end{aligned}$$

For the line integral, we have

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \oint_C M\,dx + N\,dy + L\,dz \\
&= \oint_{C'} M\,dx + N\,dy + L\,df \\
&= \oint_{C'} (M\,dx + N\,dy) + L(f_x\,dx + f_y\,dy) \\
&= \oint_{C'} (M + f_x L)\,dx + (N + f_y L)\,dy. \qquad (21)
\end{aligned}$$

More precisely, if C' is parameterized by

$$\vec{\gamma}(t) = \big(x(t), y(t)\big), \quad t \in [a,b], \qquad (22)$$

then C is parameterized by

$$\vec{r}(t) = \Big(x(t), y(t), f\big(x(t), y(t)\big)\Big), \quad t \in [a,b]. \qquad (23)$$

This means that

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \int_a^b \Big(M\big(\vec{r}(t)\big)x'(t) + N\big(\vec{r}(t)\big)y'(t) + L\big(\vec{r}(t)\big)\frac{d}{dt} f\big(x(t), y(t)\big) \Big)\, dt \\
&= \int_a^b \Big(Mx' + Ny' + L(f_x x' + f_y y') \Big)\, dt \\
&= \int_a^b \Big((M + f_x L)x' + (N + f_y L)y' \Big)\, dt \\
&= \oint_{C'} (M + f_x L)\,dx + (N + f_y L)\,dy. \qquad (24)
\end{aligned}$$

Now, using Green's Theorem, we have

$$\begin{aligned}
\oint_C \vec{F} \cdot d\vec{r} &= \oint_{C'} (M + f_x L)\,dx + (N + f_y L)\,dy \\
&= \iint_R \Big(\frac{\partial}{\partial x}(N + f_y L) - \frac{\partial}{\partial y}(M + f_x L) \Big)\, dA \\
&= \iint_R \Big(\frac{\partial}{\partial x}\big(N(x,y,f(x,y)) + f_y(x,y)L(x,y,f(x,y))\big) \\
&\qquad - \frac{\partial}{\partial y}\big(M(x,y,f(x,y)) + f_x(x,y)L(x,y,f(x,y))\big) \Big)\, dA \\
&= \iint_R \Big((N_x + N_z f_x) + \big(f_{xy} L + f_y(L_x + L_z f_x)\big) \\
&\qquad - (M_y + M_z f_y) - \big(f_{yx} L + f_x(L_y + L_z f_y)\big) \Big)\, dA \\
&= \iint_R \Big(- f_x(L_y - N_z) - f_y(M_z - L_x) + (N_x - M_y) \Big)\, dA \\
&= \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma. \qquad (25)
\end{aligned}$$

□

For the general case, we need to divide S into finitely many pieces which are graphs (under a certain projection). This includes the case where S has many boundary components like in Green's Theorem. When this happens, we add some curves to make it act like a single boundary component. We omit the proof of the general case.

Example 2. Let $\vec{F}$ be a vector field such that $\vec{\nabla} \times \vec{F} = 0$ and defined on a region containing the surface S with unit normal vector field $\hat{\mathbf{n}}$.

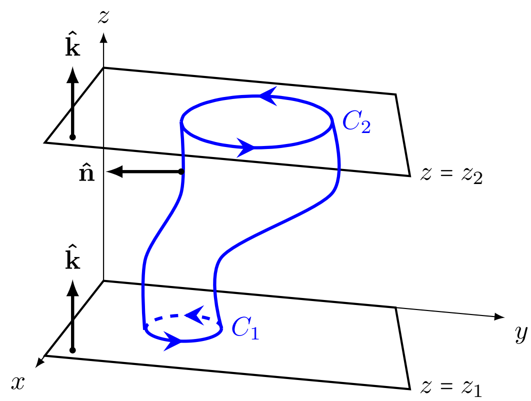

Figure 1: When $\vec{F}$ is curl-free, Stokes' Theorem gives a relationship between the circulations on the boundary components of a surface S.

The boundary C of S has two components C_1 and C_2 lying on the planes $z = z_1$ and $z = z_2$ respectively. If we orient both with respect to the unit normal $\hat{\mathbf{k}}$, then when C is oriented with respect to the surface normal $\hat{\mathbf{n}}$, we have

$$C = C_1 - C_2 \tag{26}$$

and Stokes' Theorem implies that

$$\begin{aligned} 0 = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma &= \oint_C \vec{F} \cdot d\vec{r} \\ &= \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r} \end{aligned} \tag{27}$$

and so

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_2} \vec{F} \cdot d\vec{r}. \tag{28}$$

We can compare the above example with Green's Theorem acting on a plane region with one hole:

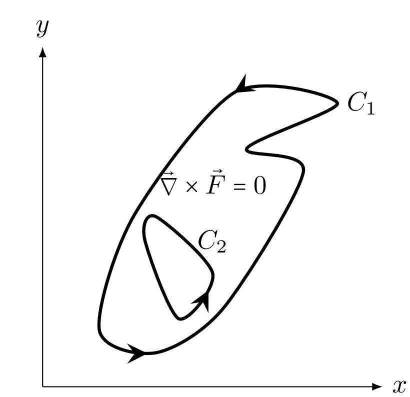

Figure 2: Green's Theorem acting on curl-free vector field $\vec{F}$ on a plane region S with one hole. Imagine pushing the upper boundary component from the previous figure down so it lies on the same plane as the lower boundary component. (You might have to deform it so that it fits inside the other curve.)

In this case, we have

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = \oint_{C_2} \vec{F} \cdot d\vec{r}. \tag{29}$$

Note which way both curves are oriented here. In particular in both cases, C_2 is oriented in the opposite direction than it would if considered as part of the boundary of S.

We now give a proof that curl-free vector fields are conservative when $n = 3$.

Proof that Curl-Free Vector Fields are Conservative ($n = 3$). By assumption we have a vector field

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}} \tag{30}$$

that is curl-free. This means that satisfies the system of equations ($\star$). That is

$$\begin{cases} \dfrac{\partial M}{\partial y} &= \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial z} &= \dfrac{\partial L}{\partial y} \\ \dfrac{\partial L}{\partial x} &= \dfrac{\partial M}{\partial z}. \end{cases} \tag{31}$$

Let C be a simple closed curve in a *simply connected* region D. Then C can be deformed to a point inside D. The process of deformation gives an oriented surface $S \subseteq D$ such that the boundary ∂S of S equals C.

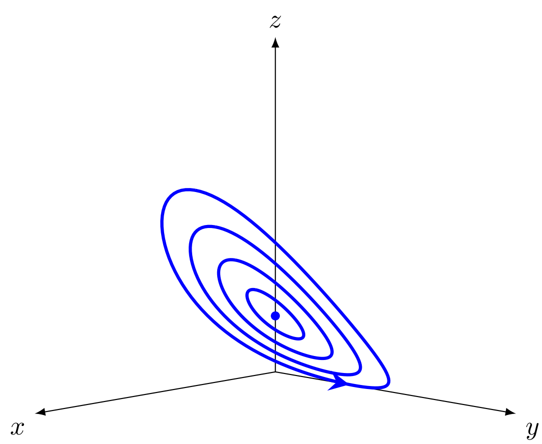

Figure 3: Contracting a curve C to a point $p \in D$ sweeps out a surface S that we can run Stokes' Theorem on.

Using Stokes' Theorem, we get

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma = 0. \tag{32}$$

Since the integral over any simple closed curve vanishes, we can conclude that $\vec{F}$ is conservative. □

In summary, we have the following table:

$n = 2$	$n = 3$
Tangential Form of Green's Theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_R (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{k}}\, dA$ $(\partial R = C)$	**Stokes' Theorem** $\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma$ $(\partial S = C)$
Normal Form of Green's Theorem $\oint_C \vec{F} \cdot \hat{\mathbf{n}}\, ds = \iint_R \vec{\nabla} \cdot \vec{F}\, dA$ $(C = \partial R)$ $\hat{\mathbf{n}}$ is the outward unit normal of the curve C in the plane	**Divergence Theorem** $\iint_S \vec{F} \cdot \hat{\mathbf{n}}\, d\sigma = \iiint_D \vec{\nabla} \cdot \vec{F}\, dV$ $(S = \partial D)$ $\hat{\mathbf{n}}$ is the outward unit normal of the surface S

0.1 Divergence Theorem

The final result of this form is the Divergence Theorem:

Theorem 3 (Divergence Theorem)**.** *Let $\vec{F}$ be a C^1 vector field on an open domain $\Omega \subseteq \mathbb{R}^3$. Let S be a* piecewise smooth oriented closed (having no boundary) surface *enclosing a (solid) region $D \subseteq \Omega$. Let $\hat{\mathbf{n}}$ be the* outward-pointing *unit normal vector field on S. Then*

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}}\, d\sigma = \iiint_D \operatorname{div} \vec{F}\, dV = \iiint_D \vec{\nabla} \cdot \vec{F}\, dV. \tag{33}$$

Example 4 (Verifying the Divergence Theorem)**.** Let S be the sphere of radius $a > 0$ centered at the origin in $\mathbb{R}^3$, that is

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = a^2\}, \tag{34}$$

and let

$$\vec{F} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}. \tag{35}$$

We note that S encloses the ball

$$B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq a^2\}. \tag{36}$$

At $(x, y, z) \in S$, the outward-pointing unit normal $\hat{\mathbf{n}}$ is given by

$$\hat{\mathbf{n}} = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{a}\big(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}\big). \tag{37}$$

We then have

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}}\, d\sigma = \iint_S \big(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}\big) \cdot \frac{1}{a}\big(x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}\big)\, d\sigma = a \iint_S d\sigma = 4\pi a^3. \tag{38}$$

On the other hand,

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = 3 \tag{39}$$

and so

$$\iiint_B \operatorname{div} \vec{F} = \iiint_B 3\, dV = 3 \cdot \left(\frac{4}{3}\pi a^3\right) = 4\pi a^3$$

as expected.

Example 5. Let

$$\vec{F} = x \sin y\,\hat{\mathbf{i}} + (\cos y + z)\,\hat{\mathbf{j}} + z^2\,\hat{\mathbf{k}}. \tag{40}$$

Compute the flux of $\vec{F}$ across the boundary ∂T of

$$T = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z \leq 1, x \geq 0, y \geq 0, z \geq 0\} \tag{41}$$

with outward-facing normal $\hat{\mathbf{n}}$.

Solution. We can use the Divergence Theorem to turn the surface integral into a volume integral.

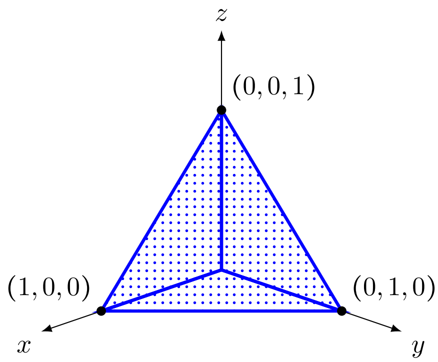

Figure 4: The tetrahedron T.

We can check that

$$\begin{aligned}\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} &= \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(\cos y + z) + \frac{\partial}{\partial z}(z^2)\\ &= \sin y - \sin y + 2z\\ &= 2z. \end{aligned} \tag{42}$$

The Divergence Theorem then says that

$$\begin{aligned}\text{Flux} &= \iint_{\partial T} \vec{F}\cdot\hat{\mathbf{n}}\,d\sigma\\ &= \iiint_T \operatorname{div}\vec{F}\,dV\\ &= \int_0^1\int_0^{1-y}\int_0^{1-x-y} 2z\,dz\,dy\,dx\\ &= \ldots = \frac{1}{12}. \qquad [\textbf{Exercise}: \text{Check this.}]\end{aligned} \tag{43}$$

□

Example 6. Let S_1 and S_2 be two surfaces with common boundary C such that $S_1 \cup S_2$ forms a closed surface enclosing a solid region D (without holes).

Suppose $\hat{\mathbf{n}}$ is the *outward-pointing* unit normal of the boundary of D. This induces orientations on S_1 and S_2. These in turn induce opposite orientations on the common boundary C since the binormals $\hat{\mathbf{b}}$ with respect to S_1 and S_2 are opposite.

Find

$$\iiint_D \operatorname{div}(\vec{\nabla}\times\vec{F})\,dV \tag{44}$$

where $\vec{F}$ is a C^2 vector field on D.

Solution. From the above discussion, we see that ∂S_1 and ∂S_2 are the same curve, but oriented in opposite directions. That is,

$$\partial S_1 = -\partial S_2 \tag{45}$$

From this and Stokes' Theorem, we have

$$\begin{aligned}\iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{\mathbf{n}}\,d\sigma &= \oint_{\partial S_1} \vec{F}\cdot d\vec{r}\\ &= -\oint_{\partial S_2} \vec{F}\cdot d\vec{r}\\ &= -\iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{\mathbf{n}}\,d\sigma.\end{aligned} \tag{46}$$

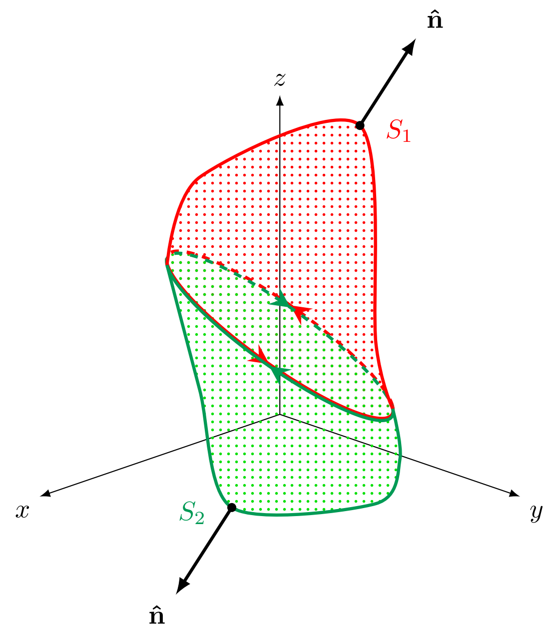

Figure 5: Two surfaces S_1 and S_2 with common boundary C enclosing a solid region D.

It follows that

$$\iint_{S_1 \cup S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma = 0. \tag{47}$$

By the Divergence Theorem, we have

$$\iiint_D \operatorname{div}(\vec{\nabla} \times \vec{F})\, dV = \iint_{S_1 \cup S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma = 0.$$

□

Remark 7. This result holds for any C^2 vector field $\vec{F}$ defined on any D. It is consistent with the fact that

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0 \tag{48}$$

for any C^2 vector field $\vec{F}$. That is,

$$\operatorname{div}(\operatorname{curl} \vec{F}) = 0. \tag{49}$$

Compare this with a similar fact that

$$\vec{\nabla} \times (\vec{\nabla} f) = 0 \tag{50}$$

or

$$\operatorname{curl}(\operatorname{grad} f) = 0 \tag{51}$$

for any C^2 function f.

(End of Lecture 19 – Nov 17)