MATH2020A Lecture 18 Notes

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Last time, we ended by taking a quick look at Stokes' Theorem. This is like a 3-dimensional version of the tangential form of Green's Theorem. In fact, Green's Theorem is a special case of Stokes' Theorem: when S is a plane surface on the xy-plane oriented with unit normal $\hat{\mathbf{k}}$.

Theorem 1 (Stokes' Theorem)**.** *Let S be a piecewise smooth oriented surface with piecewise smooth boundary C (this includes the case where C is the union of finitely many curves). Let*

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}} \tag{1}$$

be a C^1 vector field.

Suppose C is oriented counter-clockwise with respect to the unit normal $\hat{\mathbf{n}}$ on S. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{\mathbf{n}}\, d\sigma = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma. \tag{2}$$

1. If $C = C_1 \cup \ldots \cup C_k$, then it means that

$$\sum_{i=1}^{k} \oint_{C_i} \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma. \tag{3}$$

2. By C is oriented counter-clockwise with respect to $\hat{\mathbf{n}}$, we mean that we choose the direction of C such that its unit tangent vector $\hat{\mathbf{T}}$ is such that

$$\hat{\mathbf{b}} = \hat{\mathbf{n}} \times \hat{\mathbf{T}} \tag{4}$$

   points *towards* the surface S.

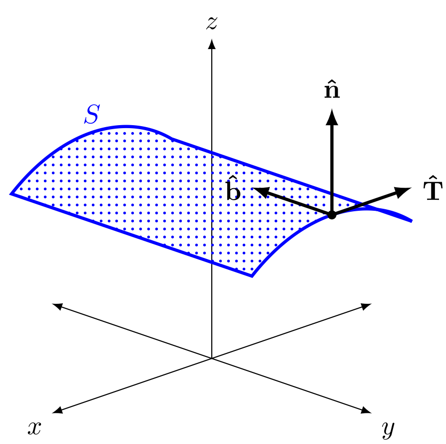

Figure 1: Orienting the boundary of a surface with respect to a choice of normal vector field $\hat{\mathbf{n}}$.

The vector $\hat{\mathbf{b}}$ is called the *unit binormal vector*. Visualizing the cross-product of two vectors in $\mathbb{R}^3$ can be done using the *right-hand rule*.

Example 2 (Orienting the Boundary of a Surface)**.** 1. Suppose $S \subseteq \mathbb{R}^2$ lies on the xy-plane is oriented with $\hat{\mathbf{n}} = \hat{\mathbf{k}}$. If its boundary ∂S is a simple closed plane curve, then to orient it with respect to our choice of $\hat{\mathbf{n}}$, we simply take the usual counter-clockwise orientation.

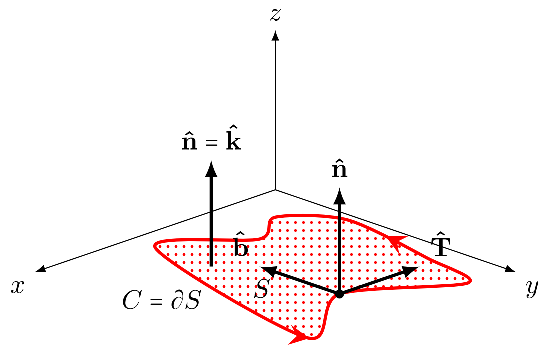

Figure 2: If S lies in the xy-plane and is bounded by a simple closed plane curve, then we have the usual counter-clockwise orientation.

2. If we have a surface with multiple boundary components, then each component will have its own orientation.

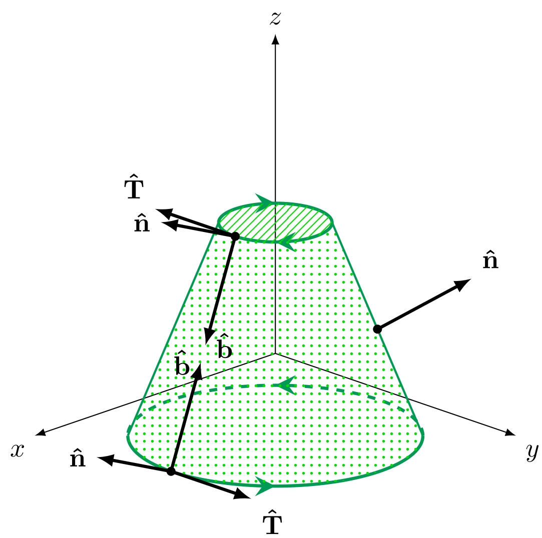

Figure 3: An oriented surface S with two (oriented) boundary components.

3. If we have a plane surface S with "holes", then to orient it with respect to $\hat{\mathbf{n}} = \hat{\mathbf{k}}$, we take the usual counter-clockwise orientation on the outer-most curve, and a clockwise orientation for the interior curves.

We can compare the situations from points 2. and 3. above and think about "deforming" the surface in 2. to the one in part 3.

Example 3 (Verifying Stokes' Theorem). 1. Let S_1 be the upper hemisphere with radius 3, that is

$$S_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 9, z \geq 0\}, \tag{5}$$

with upward pointing unit normal $\hat{\mathbf{n}}$ (with non-negative $\hat{\mathbf{k}}$-component).

We can see that the boundary of this surface is the circle

$$C\colon x^2 + y^2 = 9, \quad z = 0 \tag{6}$$

oriented counter-clockwise with respect to $\hat{\mathbf{n}}$.

Let

$$\vec{F} = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}}. \tag{7}$$

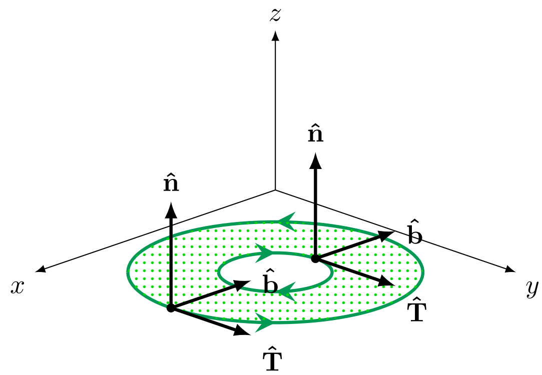

Figure 4: An oriented plane surface S with two (oriented) boundary components.

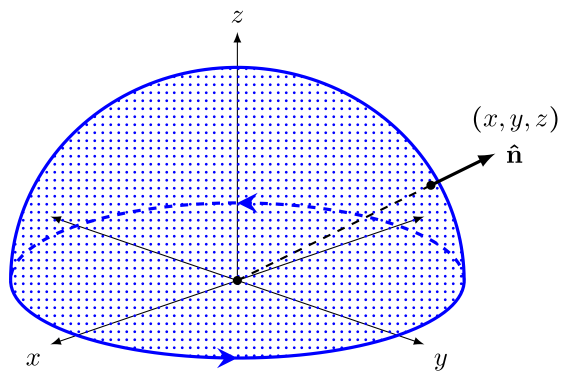

Figure 5: The upper hemisphere S_1 and a unit normal vector $\hat{\mathbf{n}}$.

We calculate both

$$\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}} \, d\sigma \text{ and } \oint_C \vec{F} \cdot d\vec{r}. \tag{8}$$

First, we can parameterize C by

$$\vec{r}(t) = 3\cos t\,\hat{\mathbf{i}} + 3\sin t\,\hat{\mathbf{j}} \qquad t \in [0, 2\pi]. \tag{9}$$

This has the correct direction with respect to the orientation $\hat{\mathbf{n}}$.

As such,

$$\begin{aligned} \oint_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \big(3\sin t\,\hat{\mathbf{i}} - 3\cos t\,\hat{\mathbf{j}}\big) \cdot \big(-3\sin t\,\hat{\mathbf{i}} + 3\cos t\,\hat{\mathbf{j}}\big)\, dt \\ &= \int_0^{2\pi} -9\, dt \\ &= -18\pi. \end{aligned} \tag{10}$$

For the surface integral, we compute that

$$\vec{\nabla} \times \vec{F} = \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{bmatrix} = -2\,\hat{\mathbf{k}}. \tag{11}$$

Since S_1 is the upper hemisphere of radius 3 centered at the origin, we can check that

$$\hat{\mathbf{n}} = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{\sqrt{x^2+y^2+z^2}} = \frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{3}. \tag{12}$$

We can consider S_1 as the graph of the function

$$z = \sqrt{9 - x^2 - y^2} \tag{13}$$

over the region

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}. \tag{14}$$

Setting

$$f(x,y) = \sqrt{9 - x^2 - y^2}, \tag{15}$$

we have

$$\vec{\nabla} f = \frac{-x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}}{\sqrt{9-x^2-y^2}} = \frac{x\,\hat{\mathbf{i}} - y\,\hat{\mathbf{j}}}{z} \tag{16}$$

and so

$$d\sigma = \sqrt{1 + \|\vec{\nabla} f\|^2}\,dx\,dy = \sqrt{1 + \frac{x^2}{z^2} + \frac{y^2}{z^2}}\,dx\,dy = \sqrt{\frac{x^2+y^2+z^2}{z^2}}\,dx\,dy = \frac{3}{z}\,dx\,dy. \tag{17}$$

Therefore

$$\begin{aligned}\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\,d\sigma &= \iint_{\{x^2+y^2\leq 9\}} \big(-2\,\hat{\mathbf{k}}\big) \cdot \left(\frac{x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}}{3}\right) \cdot \frac{3}{z}\,dx\,dy \\ &= \iint_{\{x^2+y^2\leq 9\}} -2\,dx\,dy \\ &= -18\pi.\end{aligned}$$

2. We will verify this using a new surface. Let S_2 be the disc

$$S_2 = \{(x,y,0) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 9\} \tag{18}$$

with upward pointing unit normal $\hat{\mathbf{n}} = \hat{\mathbf{k}}$.

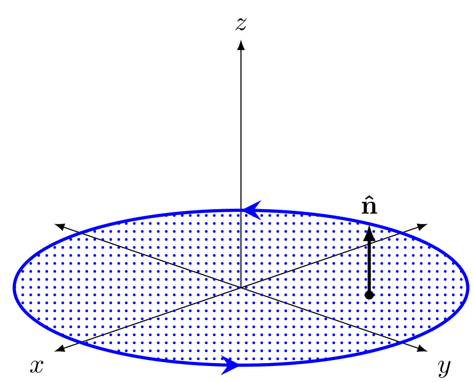

Figure 6: The disc S_2 and a unit normal vector $\hat{\mathbf{n}}$.

The boundary curve is still the same and so

$$\oint_C \vec{F} \cdot d\vec{r} = -18\pi \tag{19}$$

from before.

The surface integral in this case is

$$\iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\,d\sigma = \iint_{\{x^2+y^2\leq 9\}} (-2\hat{\mathbf{k}}) \cdot \hat{\mathbf{k}}\,dx\,dy = -18\pi. \tag{20}$$

3. Now, let $S_3 = S_1 \cup S_2$ be the surface obtained by combining the previous two. This new surface S_3 has no boundary and encloses a solid region.

   Let $\hat{\mathbf{n}}$ be the outward unit normal of the solid (see figure).

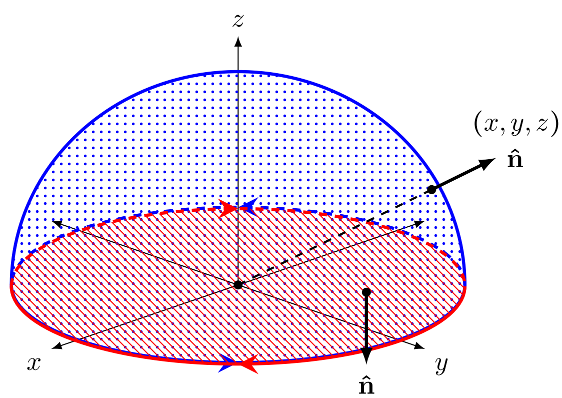

Figure 7: The surface S_3 and outward unit normal vector $\hat{\mathbf{n}}$.

We compute

$$\begin{aligned}\iint_{S_3} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma &= \iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma + \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma \\ &= -18\pi - (-18\pi) \\ &= 0. \end{aligned} \tag{21}$$

Note that in the above computation, the unit normal vector along the disc is opposite to the one we considered previously, which results in the sign change. We can also use Stokes' Theorem to consider the above as

$$\begin{aligned}\iint_{S_3} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma &= \iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma + \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma \\ &= \oint_C \vec{F} \cdot d\vec{r} - \oint_C \vec{F} \cdot d\vec{r} \\ &= 0 \end{aligned} \tag{22}$$

(End of Lecture 18 – Nov 13)