MATH2020A Lecture 17 Notes

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Last time, we discussed parameterized surfaces. We saw that the surface area of a surface S parameterized by

$$\vec{r}(u,v) = x(u,v)\,\hat{\mathbf{i}} + y(u,v)\,\hat{\mathbf{j}} + z(u,v)\,\hat{\mathbf{k}}, \quad (x,y) \in \Omega \tag{1}$$

is given by the integral

$$\text{Area}(S) = \iint_S d\sigma = \iint_\Omega \|\vec{r}_u \times \vec{r}_v\|\,dx\,dy.$$

Example 1 (Surface Area of a Graph)**.** Consider the function $z = f(x,y)$, for $(x,y) \in \Omega$. We can use the "natural" parameterization

$$\vec{r}(x,y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + f(x,y)\,\hat{\mathbf{k}}. \tag{2}$$

This means that

$$\begin{cases} \vec{r}_x &= \hat{\mathbf{i}} + f_x\,\hat{\mathbf{k}}, \\ \vec{r}_y &= \hat{\mathbf{j}} + f_y\,\hat{\mathbf{k}}. \end{cases} \tag{3}$$

Taking the cross-product, we have

$$\vec{r}_x \times \vec{r}_y = -f_x\,\hat{\mathbf{i}} + f_y\,\hat{\mathbf{j}} + \hat{\mathbf{k}}, \tag{4}$$

and so

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{1 + \|\vec{\nabla} f\|^2} \geq 1. \tag{5}$$

From this, we have the following:

Theorem 2 (Surface Area of a C^1 graph)**.** *The surface area of a C^1 graph S given by the function*

$$z = f(x,y), \qquad (x,y) \in \Omega^2 \subseteq \mathbb{R}^2 \tag{6}$$

is

$$Area(S) = \iint_\Omega \sqrt{1 + \|\vec{\nabla} f\|^2}\,dA = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2}\,dA. \tag{7}$$

Similar results hold if we had functions $x = f(y,z)$ and $y = f(x,z)$.

0.1 Level Sets (Implicit Surfaces)

Suppose we had some function $F\colon \mathbb{R}^3 \to \mathbb{R}$. We can consider sets of the form

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = c\} \tag{8}$$

where $c \in \mathbb{R}$. That is

$$S = F^{-1}(c). \tag{9}$$

In most cases, this should give us a surface. There are, of course, exceptions.

Example 3. Let

$$F(x,y,z) = x^2 + y^2 + z^2. \tag{10}$$

Is $F^{-1}(0)$ a surface?

Solution. No, it is not a surface. In this case, we see that

$$F^{-1}(0) = \{(0,0,0)\} \tag{11}$$

and so this set is just a single point, not a surface. $\square$

Remark 4 (Regular Points and Values)**.** One way to check whether we get a surface is using the gradient vector field $\vec{\nabla} F$ of F. If $\vec{\nabla} F \neq \vec{0}$ at a point p, then the Implicit Function Theorem tells us that $S = F^{-1}(c)$ is a surface near p. (In fact, it is a graph.) A point p satisfying this property is said to be a *regular point for* F.

If for some $c \in \mathbb{R}$, every point in $F^{-1}(c)$ is a regular point, then c is called a *regular value for* F (otherwise it is called a *singular value*). In this case, the entire set $F^{-1}(c)$ will be a surface.

Example 5. Returning to the previous example with

$$F(x,y,z) = x^2 + y^2 + z^2, \tag{12}$$

we can check that

$$\vec{\nabla} F = 2x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} + 2z\,\hat{\mathbf{k}}. \tag{13}$$

We see that

$$\vec{\nabla} F = \vec{0} \iff (x,y,z) = (0,0,0). \tag{14}$$

As such, any $c > 0$ is a regular value for F (note that $F^{-1}(c) = \varnothing$ when $c < 0$) and $S = F^{-1}(c)$ will be a surface for all $c > 0$.

[**Exercise** : What are these surfaces $F^{-1}(c)$ for $c > 0$?]

How can we compute the surface area of a surface $S = F^{-1}(c)$? Since $\vec{\nabla} F \neq 0$, at least one of F_x, F_y, or F_z is non-zero. Without loss of generality, assume that $F_z = \frac{\partial F}{\partial z} \neq 0$ (the other cases are similar).

The Implicit Function Theorem says that the surface

$$S = F^{-1}(c) = \{(x,y,z) \in \mathbb{R}^3 \mid F(x,y,z) = c\} \tag{15}$$

can be written locally as a graph

$$z = f(x,y) \tag{16}$$

near each point $p = (x_0, y_0, z_0) \in S$. That is

$$F\big(x, y, f(x,y)\big) = c \tag{17}$$

for all (x,y) in some domain $\Omega \subseteq \mathbb{R}^2$ near p.

By the chain rule, we have

$$\begin{cases} f_x &= -\dfrac{F_x}{F_z}, \\ f_y &= -\dfrac{F_y}{F_z}. \end{cases} \tag{18}$$

This means that locally

$$\begin{aligned} \text{Area}(S) &= \iint_\Omega \sqrt{1 + \|\vec{\nabla} f\|^2}\, dA \\ &= \iint_\Omega \sqrt{1 + \frac{F_x^2}{F_z^2} + \frac{F_y^2}{F_z^2}}\, dA \\ &= \iint_\Omega \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}\, dA. \end{aligned} \tag{19}$$

Theorem 6. *If $S = F^{-1}(c)$ is a level set such that $F_z \neq 0$ and can be represented by an implicit function* *over a domain* Ω*, then*

$$\begin{aligned} Area(S) &= \iint_\Omega \frac{\|\vec{\nabla}F\|}{|F_z|}\,dA \\ &= \iint_\Omega \frac{\|\vec{\nabla}F\|}{|F_z|}\,dx\,dy. \end{aligned} \tag{20}$$

Similar results hold when $F_x \neq 0$ or when $F_y \neq 0$.

Example 7. Find the surface area of the paraboloid

$$x^2 + y^z - z = 0 \tag{21}$$

below the plane $z = 4$. (This is, in fact, a graph, but we will do this using the method of level surfaces.)

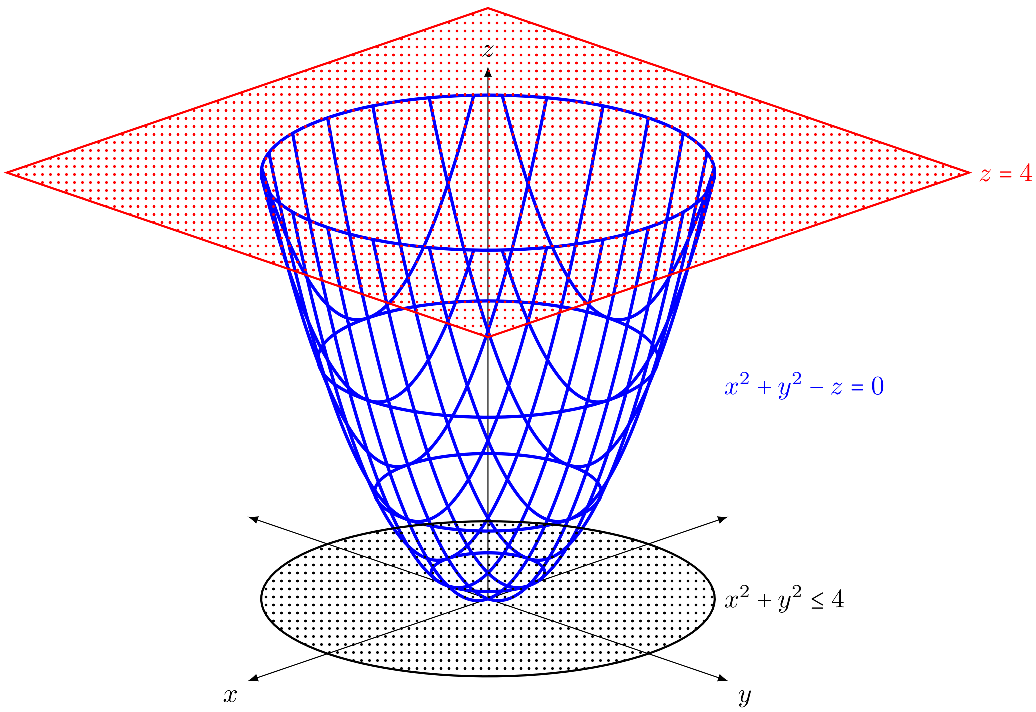

Figure 1: The paraboloid $x^2 + y^2 - z - 0$.

Solution. Let

$$F(x,y,z) = x^2 + y^2 - z. \tag{22}$$

The paraboloid intersects the plane $z = 4$ along the circle $x^2 + y^2 = 4$ and so the projected region is

$$\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}. \tag{23}$$

We can compute that

$$\vec{\nabla}F = 2x\,\hat{\mathbf{i}} + 2y\,\hat{\mathbf{j}} - \hat{\mathbf{k}}. \tag{24}$$

As such, $F_z = -1 \neq 0$ for all $(x,y) \in \Omega$.

The surface area is then given by the integral

$$\begin{aligned}
\text{Area} &= \iint_{\Omega} \frac{\|\vec{\nabla} F\|}{|F_z|}\, dA \\
&= \iint_{\Omega} \frac{\sqrt{4x^2 + 4y^2 + 1}}{|-1|}\, dx\, dy \\
&= \iint_{\{x^2+y^2 \le 4\}} \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy \\
&= \iint_{\{r \le 2\}} \sqrt{4r^2 + 1} \cdot r\, dr\, d\theta \\
&= (2\pi) \cdot \frac{1}{12} (4r^2 + 1)^{\frac{3}{2}} \Big|_{r=0}^{r=2} \\
&= \frac{\pi}{6} \big((\sqrt{17})^3 - 1 \big).
\end{aligned} \tag{25}$$

□

Definition 8 (Integral of a Function over a Surface)**.** Let $G\colon S \to \mathbb{R}$ be a *continuous* function on a surface S, parameterized by $\vec{r}(u,v)$ for $(u,v) \in R$ for some region R. The integral of G over S is

$$\iint_S G\, d\sigma := \iint_R G\big(\vec{r}(u,v)\big) \cdot \|\vec{r}_u \times \vec{r}_v\|\, dA. \tag{26}$$

Recall that $d\sigma$ is the area element of S and $dA = du\, dv$ is the area element of the parameter space.

When S is a graph $z = f(x,y)$ we have

$$\iint_S G\, d\sigma = \iint_{(x,y)} G\big(x, y, f(x,y)\big) \cdot \sqrt{1 + \|\vec{\nabla} f\|^2}\, dx\, dy. \tag{27}$$

If S is a level surface $F^{-1}(c)$ with $F_z \neq 0$, we have

$$\iint_S G\, d\sigma = \iint_{(x,y)} G(x,y,z) \cdot \frac{\|\vec{\nabla} F\|}{|F_z|}\, dx\, dy. \tag{28}$$

Example 9 (A Surface of Revolution)**.** Let S be the surface obtained by revolving the curve $y = \cos z$ for $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ about the z-axis. Let

$$G(x,y,z) = \sqrt{1 - x^2 - y^2} \tag{29}$$

be a function on S. Find

$$\iint_S G\, d\sigma. \tag{30}$$

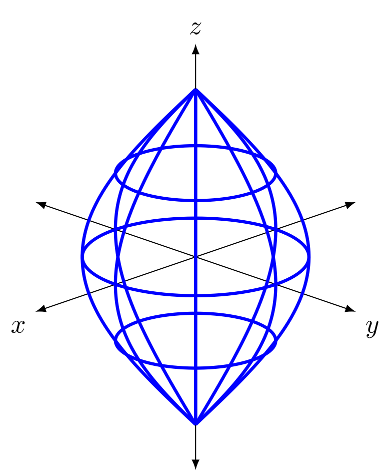

Figure 2: The surface of revolution S.

Solution. The surface S can be parameterized by

$$\begin{cases} x &= \cos z \cos\theta, \\ y &= \cos z \sin\theta, \\ z &= z. \end{cases} \tag{31}$$

for $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $\theta \in [-\pi, \pi]$. That is,

$$\vec{r}(z,\theta) = \cos z \cos\theta\,\hat{\mathbf{i}} + \cos z \sin\theta\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}. \tag{32}$$

Note here that we have a 1-dimensional *exceptional set* where $\vec{r}$ is not one-to-one when $\theta = -\pi, \pi$ or $z = -\frac{\pi}{2}, \frac{\pi}{2}$. This will not affect our calculations since this set is "small".

We can check that

$$\begin{cases} \vec{r}_z &= -\sin z \cos\theta\,\hat{\mathbf{i}} - \sin z \sin\theta, \hat{\mathbf{j}} + 1\,\hat{\mathbf{k}}, \\ \vec{r}_\theta &= -\cos z \sin\theta\,\hat{\mathbf{i}} + \cos z \cos\theta\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}}, \end{cases} \tag{33}$$

and so

$$\vec{r}_z \times \vec{r}_\theta = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\sin z \cos\theta & -\sin z \sin\theta & 1 \\ -\cos z \sin\theta & \cos z \cos\theta & 0 \end{vmatrix} = -\cos z \cos\theta\,\hat{\mathbf{i}} + \cos z \sin\theta\,\hat{\mathbf{j}} - \sin z \cos z\,\hat{\mathbf{k}}. \tag{34}$$

This means that

$$\|\vec{r}_z \times \vec{r}_\theta\| = \cos z\sqrt{1 + \sin^2 z} \tag{35}$$

(Note that $\cos z \geq 0$ since $z \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.)

It follows that

$$\begin{aligned} \iint_S G\,d\sigma &= \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} G\big(\vec{r}(z,\theta)\big)\cdot\|\vec{r}_z \times \vec{r}_\theta\|\,dz\,d\theta \\ &= \int_{-\pi}^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1-\cos^2 z}\cdot\cos z\sqrt{1+\sin^2 z}\,dz\,d\theta \\ &= (2\pi)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin z \cos z\sqrt{1+\sin^2 z}\,dz \\ &= \ldots = \frac{4\pi}{3}(2\sqrt{2}-1). \qquad [\textbf{Exercise}: \text{Check this.}] \end{aligned} \tag{36}$$

□

0.2 Orientation of Surfaces

Recall that we required an orientation on a curve to integrate vector fields along them. In the same vein, we require orientations on surfaces to integrate vector fields.

Definition 10 (Orientation of a Surface in $\mathbb{R}^3$)**.** A surface S is *orientable* if we can define a *unit normal* vector *continuously* at every point of S. Such a chosen normal vector field is called an *orientation* of S.

Example 11. 1. Consider the 2-sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}. \tag{37}$$

The vector field

$$\hat{\mathbf{n}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \tag{38}$$

is continuous and has

$$\|\hat{\mathbf{n}}\| = \sqrt{x^2 + y^2 + z^2} = 1 \tag{39}$$

(which justifies the use of the hat notation).

As such, S^2 is orientable.

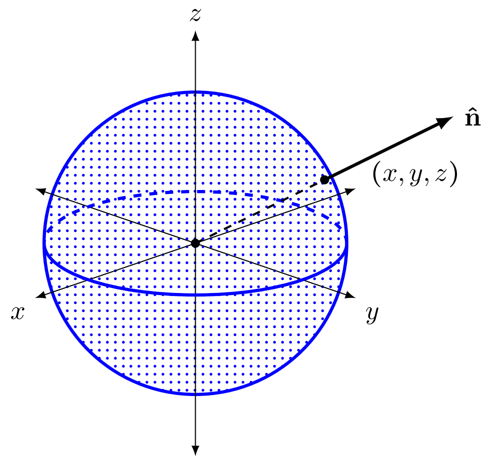

Figure 3: The 2-sphere S^2 and a unit normal vector $\hat{\mathbf{n}}$.

2. The torus T^2 is orientable.

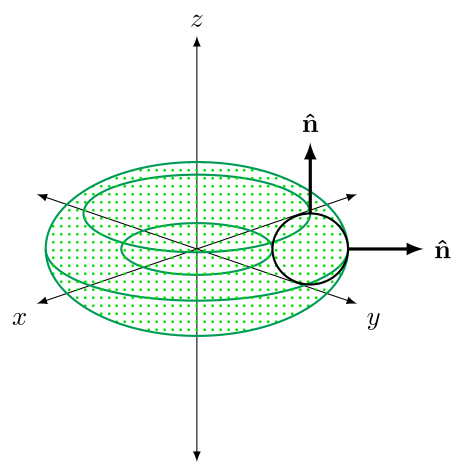

Figure 4: The torus T^2 and unit normal vectors $\hat{\mathbf{n}}$.

3. The Möbius strip is *not* orientable. It is sometimes referred to as a one-sided surface. You can make a model of a Möbius strip by taking a strip of paper and twisting it once before attaching the ends back together.

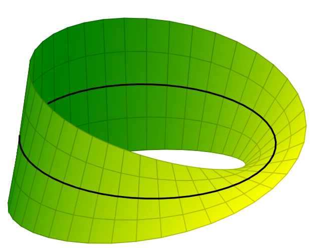

Figure 5: The Möbius strip is an example of a non-orientable surface.

4. The Klein bottle is another example of a *non-orientable surface*. This can be constructed by taking a cylinder, inverting one end, then attaching the ends (this may require puncturing the original cylinder to achieve.)

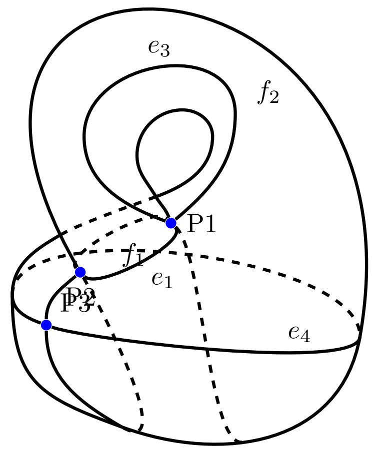

Figure 6: The Klein bottle is another non-orientable surface.

Remark 12. Parametric surfaces $S = \vec{r}(u,v)$ are always orientable. The unit normal vector field

$$\hat{\mathbf{n}} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} \tag{40}$$

given by the parameterization is a continuous unit normal vector field on S.

Given a connected orientable surface $S \subseteq \mathbb{R}^3$, there are two ways to assign the unit normal vector field $\hat{\mathbf{n}}$.

Definition 13 (Compatible Parameterizations and Orientations)**.** Suppose S is orientable and we have already chosen a continuous unit normal vector field $\hat{\mathbf{n}}$. We say that a parameterization $\vec{r}(u,v)$ of S is *compatible* with the orientation $\hat{\mathbf{n}}$ if

$$\hat{\mathbf{n}} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}. \tag{41}$$

Definition 14 (Flux through a Surface)**.** Let S be a surface with orientation $\hat{\mathbf{n}}$. Let $\vec{F}$ be a vector field on S. The *flux* of $\vec{F}$ across S is

$$\text{Flux} = \iint_S \vec{F} \cdot \hat{\mathbf{n}}\, d\sigma. \tag{42}$$

Example 15. Let S be the surface given by

$$y = x^2, \quad 0 \le x \le 1, \quad 0 \le z \le 4. \tag{43}$$

with $\hat{\mathbf{n}}$ given by the natural parameterization

$$\vec{r}(x,z) = x\,\hat{\mathbf{i}} + x^2\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}. \tag{44}$$

Let

$$\vec{F} = yz\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} - z^2\,\hat{\mathbf{k}}. \tag{45}$$

Find

$$\iint_S \vec{F} \cdot \hat{\mathbf{n}}\, d\sigma. \tag{46}$$

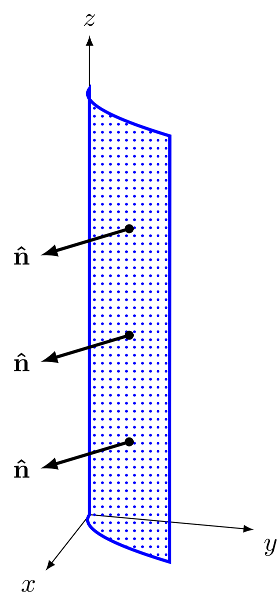

Figure 7: The surface S.

Solution. To calculate

$$\hat{\mathbf{n}} = \frac{\vec{r}_x \times \vec{r}_z}{\|\vec{r}_x \times \vec{r}_z\|} \tag{47}$$

we have

$$\begin{cases} \vec{r}_x &= \hat{\mathbf{i}} + 2x\,\hat{\mathbf{j}}, \\ \vec{r}_z &= \hat{\mathbf{k}}. \end{cases} \tag{48}$$

Thus

$$\vec{r}_x \times \vec{r}_z = 2x\,\hat{\mathbf{i}} - \hat{\mathbf{j}}. \tag{49}$$

This means that

$$\hat{\mathbf{n}} = \frac{2x\,\hat{\mathbf{i}} - \hat{\mathbf{j}}}{\sqrt{4x^2+1}}. \tag{50}$$

Then

$$\begin{aligned} \text{Flux} &= \iint_S \vec{F} \cdot \hat{\mathbf{n}}\, d\sigma \\ &= \int_0^4 \int_0^1 \left(yz\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} - z^2\,\hat{\mathbf{k}}\right) \cdot \left(\frac{2x\,\hat{\mathbf{i}} - \hat{\mathbf{j}}}{\sqrt{4x+1}}\right) \cdot \sqrt{4x^2+1}\, dx\, dz \\ &= \int_0^4 \int_0^1 (2x^3 z - x)\, dx\, dz \\ &= 2. \end{aligned} \tag{51}$$

□

Remark 16. One thing we can note from the previous example is that the factors of $\|\vec{r}_x \times \vec{r}_z\|$ from $\hat{\mathbf{n}}$ and $d\sigma$ cancel. This allows us to rewrite the expression (using generic (u, v) coordinates) as

$$\text{Flux} = \iint_{(u,v)} \vec{F}\big(\vec{r}(u,v)\big) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv. \tag{52}$$

Here

$$\hat{\mathbf{n}} \cdot d\sigma = (\vec{r}_u \times \vec{r}_v)\, du\, dv \tag{53}$$

is the *oriented area element*.

The next result allows us to to relate our previous idea of flux through a closed curve with our recent notion of flux.

Theorem 17 (Stokes' Theorem)**.** *Let S be a piecewise smooth oriented surface with piecewise smooth boundary C (this includes the case where C is the union of finitely many curves). Let*

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}} \tag{54}$$

be a C^1 vector field.

Suppose C is oriented counter-clockwise with respect to the unit normal $\hat{\mathbf{n}}$ on S. Then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{\mathbf{n}}\, d\sigma = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{n}}\, d\sigma. \tag{55}$$

We can compare this to our earlier result of (the tangential form of) Green's Theorem:

Theorem 18 (Green's Theorem)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open and*

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} \tag{56}$$

be a C^1 vector field on Ω. If C is a piecewise smooth simple closed counter-clockwise oriented curve enclosing a region R lying entirely in Ω, then

$$[\textit{Normal Form}] \qquad \oint_C \vec{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C M\, dy - N\, dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy, \tag{57}$$

$$[\textit{Tangential Form}] \qquad \oint_C \vec{F} \cdot \hat{\mathbf{T}}\, ds = \oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy. \tag{58}$$

The big important thing here is that without the notion of an orientation in $\mathbb{R}^3$, there is no sense of "counter-clockwise oriented". In our previous discussions using Green's Theorem, we implicitly took the vector $\hat{\mathbf{k}}$ to be our orientation. This is somewhat natural to do since we usually picture $\mathbb{R}^3$ with the xy-plane being the "floor".

(End of Lecture 17 – Nov 10)