MATH2020A Lecture 16 Notes

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Last time, we introduced some differential operators. These will be used quite often throughout.

Remark 1 (Generalizations of Divergence, Curl, and Cross Products)**.** The *divergence* operator can be defined on $\mathbb{R}^n$ for any n.

Explicitly, in $\Omega \subseteq \mathbb{R}^3$, the *divergence* of a vector field

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}} \tag{1}$$

is defined to be

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}\,\hat{\mathbf{j}} = \frac{\partial}{\partial z}\,\hat{\mathbf{k}}\right)\cdot\left(M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}}\right) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial L}{\partial z}. \tag{2}$$

The *curl* operator can only be defined in certain dimensions because it involves the cross product. It is an algebraic fact that vector cross products only exist in dimensions 0, 1, 3, and 7 (which is why the definition above involves $\hat{\mathbf{k}}$ even though $\vec{F}$ was on $\mathbb{R}^2$).

These correspond to the *imaginary parts* of the *normed division algebras* of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$, and the octonions $\mathbb{O}$ respectively.

The curl of a vector field

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}}$$

on $\Omega \subseteq \mathbb{R}^3$ is given by

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \left(\frac{\partial L}{\partial y} - \frac{\partial N}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial M}{\partial z} - \frac{\partial L}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{\mathbf{k}}.$$

The divergence "measures" how much a vector field $\vec{F}$ moves away from the origin.

The rotation "measures" how much a vector field spins counter-clockwise.

The curl is the 3 dimensional version of the rotation. It quantifies how much it "spins" about the three coordinate axes.

We have the following [**Exercise** : Check this]:

i) $\vec{\nabla}\times(\vec{\nabla}f) = \vec{0}$ (*i.e.* $\operatorname{curl}(\operatorname{grad} f) = 0$),

ii) if $\vec{F}$ is conservative, then $\vec{\nabla}\times\vec{F} = \operatorname{curl}\vec{F} = 0$,

iii) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$ (*i.e.* $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$).

Remark 2 (Laplacian)**.** In general, for a function f we have

$$\vec{\nabla}\cdot(\vec{\nabla}f) = \operatorname{div}(\operatorname{grad} f) \neq 0. \tag{3}$$

This quantity is called the *Laplacian* of f and is denoted by

$$\vec{\nabla}^2 f = \vec{\nabla}\cdot(\vec{\nabla}f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}. \tag{4}$$

This generalizes to any dimension n.

Another common notation for this concept is

$$\Delta = \vec{\nabla}^2 \text{ or } \Delta = -\vec{\nabla}^2 \tag{5}$$

(The difference is sign is based on convention and different sub-fields of math use different ones.)

The *operator* $\vec{\nabla}^2$ is called the *Laplace operator* and the equation

$$\vec{\nabla}^2 f = 0 \tag{6}$$

is called the *Laplace equation*. Solution to the *Laplace equation* are called *harmonic functions*.

Proof that Curl-Free Vector Fields are Conservative $(n = 2)$. Suppose

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} \tag{7}$$

is a vector field on an open, (path) connected, simply connected domain $\Omega \subseteq \mathbb{R}^2$ such that

$$\vec{\nabla} \times \vec{F} = \operatorname{curl} \vec{F} = 0. \tag{8}$$

As such, we have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{9}$$

Let C_1 and C_2 be oriented curves in Ω with the same start point and end point. We have two cases:

- (Case 1: C_1 and C_2 do not intersect)

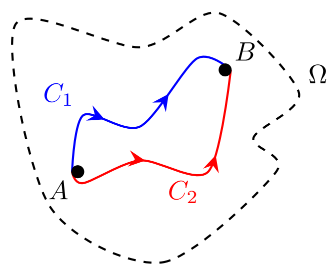

Figure 1: Two non-intersecting curves C_1 and C_2 will enclose a simply connected region R.

  In this case, since Ω is simply connected, so is the region R bounded by C_1 and C_2. By Green's Theorem, we have

$$0 = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \pm \oint_{C_1 - C_2} M\,dx + N\,dy. \tag{10}$$

  Hence,

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} M\,dx + N\,dy = \int_{C_2} M\,dx + N\,dy = \int_{C_2} \vec{F} \cdot d\vec{r}. \tag{11}$$

- (Case 2: C_1 and C_2 intersect somewhere)

  In this case, we pick a new curve C_3 with the same start and end points that does not intersect either C_1 or C_2. [**Exercise** : Why is this possible?]

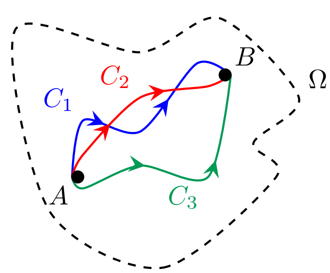

Figure 2: Picking a new curve C_3 that does not intersect with either C_1 or C_2.

The previous case tells us that

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r} \text{ and } \int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_3} \vec{F} \cdot d\vec{r}. \tag{12}$$

Putting these together, we get

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \tag{13}$$

as desired.

□

In order to apply Green's Theorem to more general situations, we have

Theorem 3 (Green's Theorem (General Form))**.** *Suppose that C is a piecewise smooth simple closed (counter-clockwise oriented) curve in $\mathbb{R}^2$. Suppose also that $C_1, C_2, \ldots, C_m$ are pairwise disjoint, piecewise smooth simple closed (counter-clockwise oriented) curves each enclosed by C.*

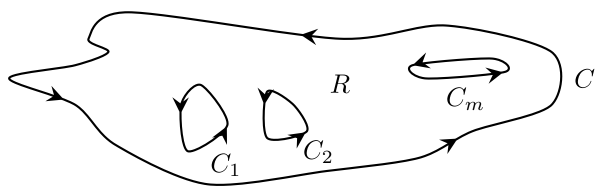

Figure 3: The general set-up for Green's Theorem.

Let R be the region between C and $C_1, C_2, \ldots, C_m$ and suppose that

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} \tag{14}$$

is C^1 and is defined on some open set Ω containing R. Then

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \oint_C M\,dx + N\,dy - \sum_{k=1}^{m} \oint_{C_k} M\,dx + N\,dy. \tag{15}$$

(This is the tangential form of Green's Theorem, the normal form is similar.)

Sketch of Proof. For simplicity, we will assume that there is only one piecewise smooth simple closed curve C_1 enclosed by C.

We begin by connecting C and C_1 using an arc L. This allows us to define a new closed curve

$$C^* = C + L - C_1 - L. \tag{16}$$

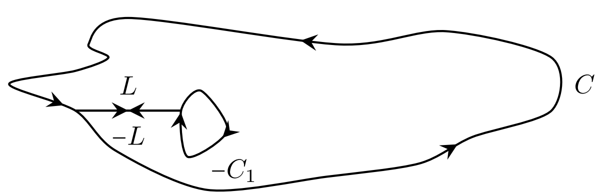

Figure 4: Creating C^* by connecting C and C_1 with an arc L.

The region R enclosed by C and C_1 is also the region enclosed by C^* except the section along the arc L.

The section along L is "small" and so

$$\begin{aligned}
\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA &= \iint_{R\setminus L} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \\
&= \oint_{C^*} (M\,dx + N\,dy) \\
&= \oint_C (M\,dx + N\,dy) + \int_L (M\,dx + N\,dy) - \oint_{C_1} (M\,dx + N\,dy) - \int_L (M\,dx + N\,dy) \\
&= \oint_C (M\,dx + N\,dy) - \oint_{C_1} (M\,dx + N\,dy). \qquad (17)
\end{aligned}$$

□

Example 4. Let

$$\vec{F} = -\frac{y}{x^2+y^2}\,\hat{\mathbf{i}} + \frac{x}{x^2+y^2}\,\hat{\mathbf{j}} \text{ on } \Omega = \mathbb{R}^2 \setminus \{(0,0)\}. \qquad (18)$$

We saw from before that

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = 2\pi$$

where C_1 is the unit circle oriented clockwise.

Consider now the oriented curves C and C' shown below (we have plotted C_1 in red for reference)

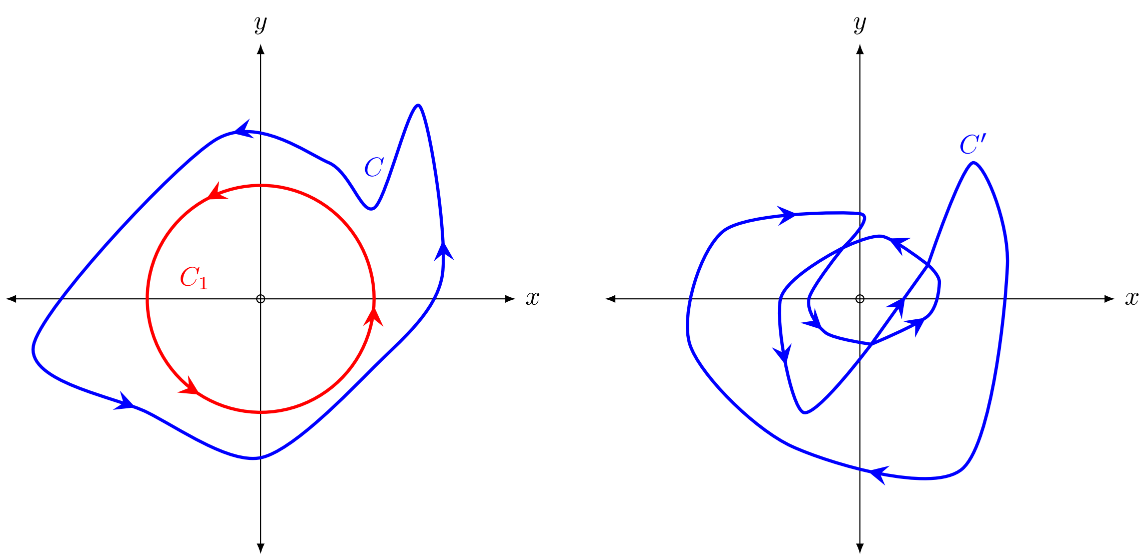

Figure 5: Two curves C and C'.

What are

$$\oint_C \vec{F} \cdot d\vec{r} \text{ and } \oint_{C'} \vec{F} \cdot d\vec{r}? \qquad (19)$$

Solution. We first note that the original statement Green's Theorem does not apply directly since C encloses the origin $(0,0)$, where $\vec{F}$ is not defined.

We can check that $\vec{F}$ is curl-free:

$$\begin{aligned}
\vec{\nabla} \times \vec{F} &= \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right]\hat{\mathbf{k}} \\
&= \left[\left(\frac{x(2x) - (x^2+y^2)}{(x^2+y^2)^2}\right) + \left(\frac{y(2y) - (x^2+y^2)}{(x^2+y^2)^2}\right)\right]\hat{\mathbf{k}} \\
&= \vec{0}. \qquad (20)
\end{aligned}$$

Using the general form of Green's Theorem (Theorem 3), we have

$$\oint_C \vec{F} \cdot d\vec{r} - \oint_{C_1} \vec{F} \cdot d\vec{r} = \iint_R (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{k}} \, dA = 0. \tag{21}$$

This means that

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} = 2\pi. \tag{22}$$

[**Exercise** : Use a similar argument to show that

$$\oint_C \vec{F} \cdot d\vec{r} = 2\pi$$

for any (counter-clockwise oriented) simple closed curve C enclosing the origin.]

Remark 5 (Winding Number)**.** This result relates to a concept called the winding number of a curve. If we take a closed curve C that encloses the origin, but goes around it multiple times, we would expect the integral

$$\oint_C \vec{F} \cdot d\vec{r}$$

to be some integer multiple of 2π.

In particular, taking the integral detects exactly how many times this goes around the origin.

For the second curve C', we see that we can split it into two parts

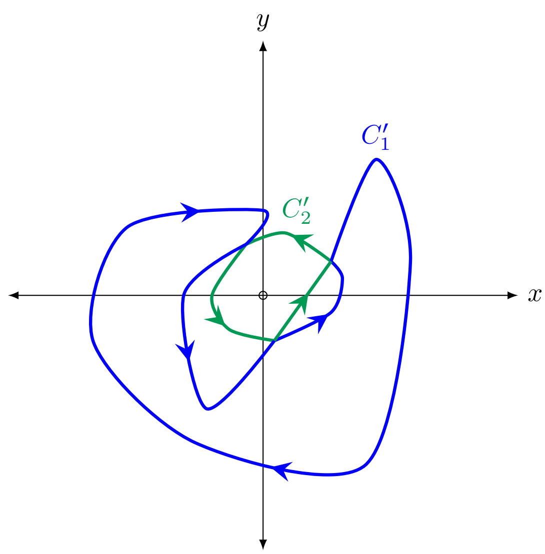

Figure 6: Splitting C' into two simple closed curves C_1' and C_2'.

The first one C_1' encloses a region R_1 that does not contain the origin, while the second one C_2' encloses one that does.

By Green's Theorem,

$$\oint_{C_1'} \vec{F} \cdot d\vec{r} = \iint_{R_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{k}} \, dA = 0, \tag{23}$$

and by the previous part, we have

$$\oint_{C_2'} \vec{F} \cdot d\vec{r} = 2\pi. \tag{24}$$

Hence

$$\oint_{C'} \vec{F} \cdot d\vec{r} = 0 + 2\pi = 2\pi. \tag{25}$$

□

Remark 6 (Homotopy)**.** Another idea here is that of homotopy. This is a process of continually deforming a curve into another. In the case of C', we can push it around in a way that doesn't cross the origin until it turns into a closed curve that wraps around the origin once.

The idea of splitting curves into smaller, more manageable ones can be quite helpful. Consider another complicated curve C as shown below:

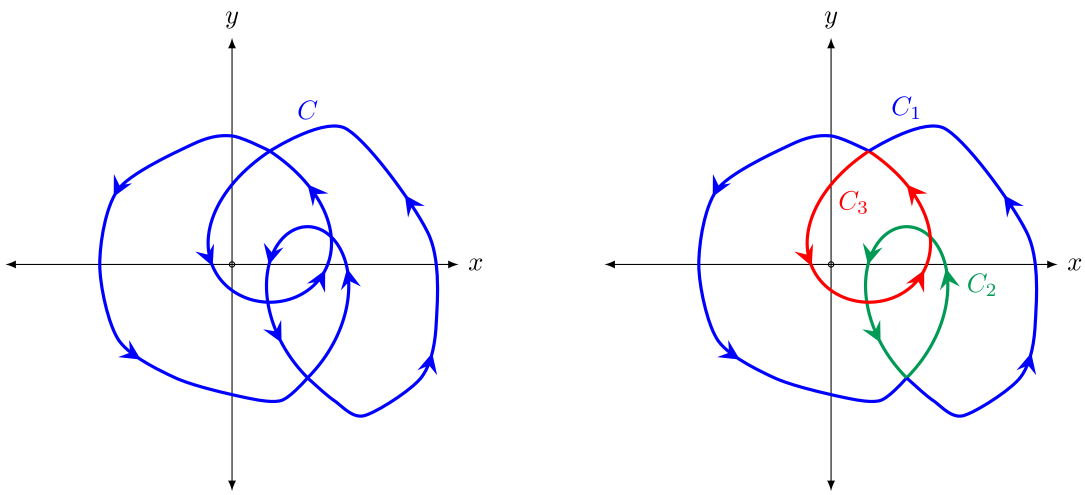

Figure 7: Decomposing another complicated curve C into sub-curves C_1, C_2, and C_3.

We can split it into smaller curves, both C_1 and C_3 wrap around the origin counter-clockwise once, while C_2 does not at all.

Together, we see that

$$\oint_C \vec{F} \cdot d\vec{r} = 2\pi + 0 + 2\pi = 4\pi. \tag{26}$$

0.1 Surface Areas and Integrals

Definition 7 (Parametric Surfaces)**.** A *parametric surface* (or *parametrization of a surface*) in $\mathbb{R}^3$ is a *continuous* mapping of 2 variables into $\mathbb{R}^3$:

$$\vec{r}(u,v) = x(u,v)\,\hat{\mathbf{i}} + y(u,v)\,\hat{\mathbf{j}} + z(u,v)\,\hat{\mathbf{k}}. \tag{27}$$

It is called *smooth* if

- $\vec{r}$ is C^1, that is the partial derivatives x_u, x_v, y_u, y_v, z_u, and z_v are continuous.
- $\vec{r}_u \times \vec{r}_v \neq 0$ for all u, v, where

$$\begin{cases} \vec{r}_u &= \dfrac{\partial \vec{r}}{\partial u} = \dfrac{\partial x}{\partial u}\,\hat{\mathbf{i}} + \dfrac{\partial y}{\partial u}\,\hat{\mathbf{j}} + \dfrac{\partial z}{\partial u}\,\hat{\mathbf{k}}, \\ \vec{r}_v &= \dfrac{\partial \vec{r}}{\partial v} = \dfrac{\partial x}{\partial v}\,\hat{\mathbf{i}} + \dfrac{\partial y}{\partial v}\,\hat{\mathbf{j}} + \dfrac{\partial z}{\partial v}\,\hat{\mathbf{k}}. \end{cases} \tag{28}$$

Remark 8 (Non-Degeneracy)**.** Note that the second condition implies that $\vec{r}_u$ and $\vec{r}_v$ are linearly independent and so $\operatorname{span}\{\vec{r}_u, \vec{r}_v\}$ is in fact a 2-dimensional space.

In particular, a surface does not degenerate to a curve or a point.

Example 9 (Torus)**.** Consider the circle on the xz-plane (so $y = 0$) with radius $a > 0$ centered at $(x,z) = (R, 0)$ with $R > a$.

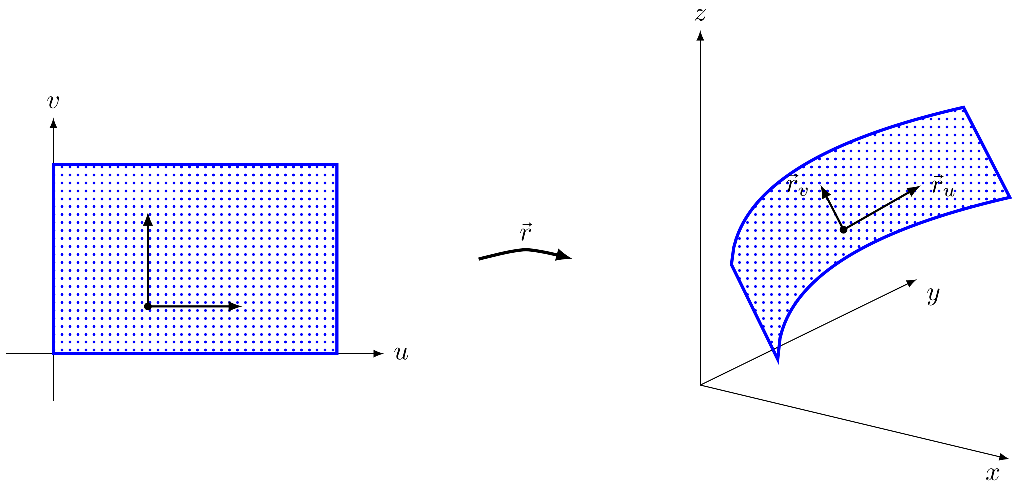

Figure 8: Non-degeneracy of a surface.

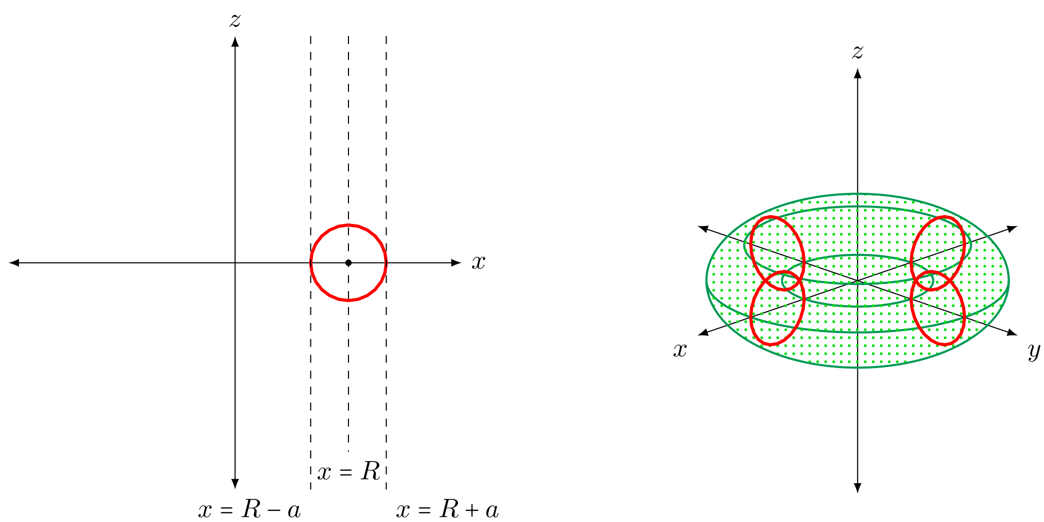

Figure 9: Constructing a torus by rotating a circle in the xz-plane about the z-axis.

A parameterization of this circle can be given by

$$\begin{cases} x &= R + a\cos\alpha, \\ z &= a\sin\alpha, \end{cases} \qquad \alpha \in [0, 2\pi]. \tag{29}$$

By rotating this about the z-axis, we get the torus. This can be parameterized by

$$\begin{cases} x &= (R + a\cos\alpha)\cos\theta, \\ y &= (R + a\cos\alpha)\sin\theta, \\ z &= a\sin\alpha, \end{cases} \qquad \alpha \in [0, 2\pi], \quad \theta \in [0, 2\pi]. \tag{30}$$

[**Exercise** : Check that this is smooth.)]

Remark 10. This torus can also be described as the set where

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = a^2. \tag{31}$$

[**Exercise** : Check this.]

0.1.1 Surface Area

Recall that for $\vec{a}, \vec{b} \in \mathbb{R}^3$ that the area of the parallelogram generated by $\vec{a}$ and $\vec{b}$ is $\|\vec{a} \times \vec{b}\|$.

Let $\vec{r}(u,v)$ be a parameterization of a surface S with $(u,v) \in \Omega$ for some domain Ω.

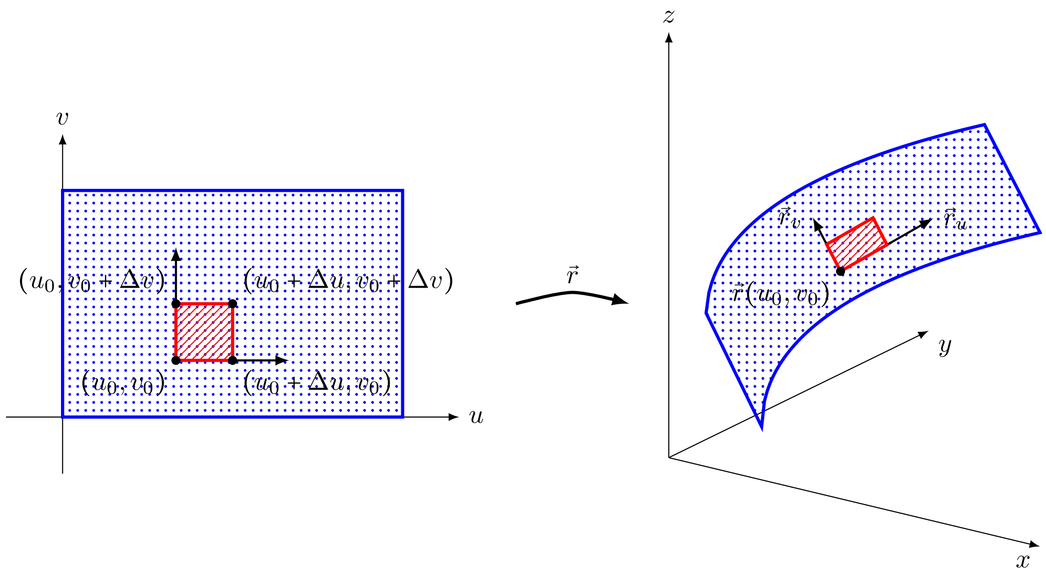

Figure 10: The parameterization $\vec{r}$ causes a change in infinitesimal area.

Under the parameterization $\vec{r}$, a rectangle based at (u_0, v_0) in the domain Ω gets mapped to some set on the surface S (see related figure). This new set can be approximated by a parallelogram based at $\vec{r}(u_0, v_0)$ generated by the vectors $\Delta u \cdot \vec{r}_u$ and $\Delta v \cdot \vec{r}_v$. (This parallelogram lies on an object called the tangent plane.)

Using the formula from earlier, this approximate parallelogram has area

$$\|(\Delta u \cdot \vec{r}_u) \times (\Delta v \cdot \vec{r}_v)\| = (\Delta u)(\Delta v) \cdot \|\vec{r}_u \times \vec{r}_v\|. \tag{32}$$

As such, the *area element of* S, denoted $d\sigma$, is given by

$$d\sigma = \|\vec{r}_u \times \vec{r}_v\|\, du\, dv = \|\vec{r}_u \times \vec{r}_v\|\, dA. \tag{33}$$

As such, we have the following:

Definition 11 (Surface Area)**.** Let $S \subseteq \mathbb{R}^3$ be a *smooth parameteric surface* given by $\vec{r}(u,v)$ for $(u,v) \in \Omega \subseteq \mathbb{R}^2$. Then

$$\text{Area}(S) \coloneqq \iint_\Omega d\sigma = \iint_\Omega \|\vec{r}_u \times \vec{r}_v\|\, dA = \iint_\Omega \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA \tag{34}$$

Example 12 (Surface Area of a Torus)**.** Recall from Example 9 that we could parameterize a torus by

$$\begin{cases} x &= (R + a\cos\alpha)\cos\theta, \\ y &= (R + a\cos\alpha)\sin\theta, \\ z &= a\sin\alpha, \end{cases} \qquad \alpha \in [0, 2\pi], \quad \theta \in [0, 2\pi]. \tag{35}$$

Solution. Using the parameterization

$$\vec{r}(\alpha, \theta) = (R + a\cos\alpha)\cos\theta\,\hat{\mathbf{i}} + (R + a\cos\alpha)\sin\theta\,\hat{\mathbf{j}} + a\sin\alpha\,\hat{\mathbf{k}}, \tag{36}$$

we see that

$$\begin{cases} \vec{r}_\alpha &= -a\sin\alpha\cos\theta\,\hat{\mathbf{i}} - a\sin\alpha\sin\theta\,\hat{\mathbf{j}} + a\cos\alpha\,\hat{\mathbf{k}}, \\ \vec{r}_\theta &= -(R + a\cos\alpha)\sin\theta\,\hat{\mathbf{i}} + (R + a\cos\alpha)\cos\theta\,\hat{\mathbf{j}}. \end{cases} \tag{37}$$

Taking the cross-product, we get

$$\vec{r}_\alpha \times \vec{r}_\theta = -a(r + a\cos\alpha)\cos\alpha\cos\theta\,\hat{\mathbf{i}} - a(R + a\cos\alpha)\cos\alpha\sin\theta\,\hat{\mathbf{j}} - a(R + a\cos\alpha)\sin\alpha\,\hat{\mathbf{k}}. \tag{38}$$

[**Exercise** : Check this.]

This means that

$$\|\vec{r}_\alpha \times \vec{r}_\theta\| = a(R + a\cos\alpha) > 0, \tag{39}$$

[**Exercise** : Check this.] (and we see that the surface is *smooth*).

We can then compute that the surface area is given by

$$\begin{aligned} \text{Area(Torus)} &= \iint_\Omega \|\vec{r}_\alpha \times \vec{r}_\theta\|\,dA \\ &= \int_0^{2\pi}\int_0^{2\pi} a(R + a\cos\alpha)\,d\alpha\,d\theta \\ &= 4\pi^2 Ra. \qquad [\textbf{Exercise}: \text{Check this.}] \end{aligned} \tag{40}$$

□

Note that it is not a coincidence that this area is the product of the areas of the individual circles.

(End of Lecture 16 – Nov 3)