MATH2020A Lecture 15 Notes

Caleb Suan

Last time, we were in the middle of a proof of Green's Theorem for simple domains. Recall that a simple domain in $\mathbb{R}^2$ is one that can be written as both a Type I domain and a Type II domain.

Theorem 1 (Green's Theorem)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open and*

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} \tag{1}$$

be a C^1 vector field on Ω. If C is a piecewise smooth simple closed counter-clockwise oriented curve enclosing a region R lying entirely in Ω, then

$$[\text{Normal Form}] \qquad \oint_C \vec{F} \cdot \hat{\mathbf{n}}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\,dy, \tag{2}$$

$$[\text{Tangential Form}] \qquad \oint_C \vec{F} \cdot \hat{\mathbf{T}}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy. \tag{3}$$

Proof of Green's Theorem for Simple Domains. Let R be a simple domain, as such R is Type I and we can write

$$R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}. \tag{4}$$

Denote the four components of the boundary ∂R of R by the curves C_1, C_2, C_3, and C_4 as in the figure. (Note that it is possible for C_2 or C_4 to be a point.) We then have

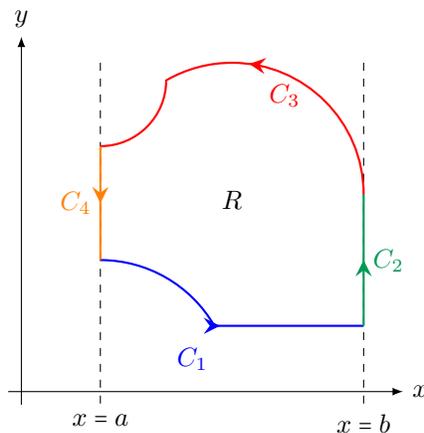$$\partial R = C_1 + C_2 + C_3 + C_4 \tag{5}$$

as oriented curves.

Figure 1: Realizing R as a Type I domain and labelling parts of its boundary ∂R.

We can parameterize

$$C_1 = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, y = g_1(x)\} \tag{6}$$

by

$$\vec{r_1}(t) = \big(t, g_1(t)\big), \qquad t \in [a,b]. \tag{7}$$

Likewise, we can parameterize $-C_3$ by

$$\vec{r}_3(t) = \big(t, g_2(t)\big), \qquad t \in [a, b]. \tag{8}$$

Note that we have to ensure that the parameterizations give the proper orientations of the curves.

Hence

$$\int_{C_1} M \, dx = \int_a^b M\big(t, g_1(t)\big) \, dt \quad \text{and} \quad \int_{-C_3} M \, dx = \int_a^b M\big(t, g_2(t)\big) \, dt. \tag{9}$$

For the vertical segments, we have

$$C_2 = \{(x, y) \in \mathbb{R}^2 \mid x = b, g_1(b) \le y \le g_2(b)\} \tag{10}$$

and so we can parameterize is by

$$\vec{r}_2(t) = (b, t), \qquad t \in [g_1(b), g_2(b)]. \tag{11}$$

Similarly, we can parameterize $-C_4$ by

$$\vec{r}_4(t) = (a, t), \qquad t \in [g_1(a), g_2(a)]. \tag{12}$$

Hence

$$\int_{C_2} M \, dx = \int_{g_1(b)}^{g_2(b)} M(b, t) \, db = 0 \quad \text{and} \quad \int_{-C_4} M \, dx = \int_{g_1(a)}^{g_2(a)} M(a, t) \, da = 0. \tag{13}$$

Putting everything together, we get

$$\oint_{\partial R} M \, dx = \sum_{k=1}^4 \int_{C_k} M \, dx = \int_a^b \Big(M\big(t, g_1(t)\big) - M\big(t, g_2(t)\big) \Big) \, dt. \tag{14}$$

On the other hand, using Fubini's Theorem, we have

$$\begin{aligned} \iint_R \left(-\frac{\partial M}{\partial y} \right) dA &= \int_a^b \int_{g_1(x)}^{g_2(x)} \left(-\frac{\partial M}{\partial y} \right) dy \, dx \\ &= \int_a^b - \Big(M\big(x, g_2(x)\big) - M\big(x, g_1(x)\big) \Big) \, dx \\ &= \oint_{\partial R} M \, dx. \end{aligned} \tag{15}$$

Now, since R is also of Type II, we can do a similar thing and write

$$R = \{(x, y) \in \mathbb{R}^2 \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}, \tag{16}$$

and

$$\partial R = C_1' + C_2' + C_3' + C_4'. \tag{17}$$

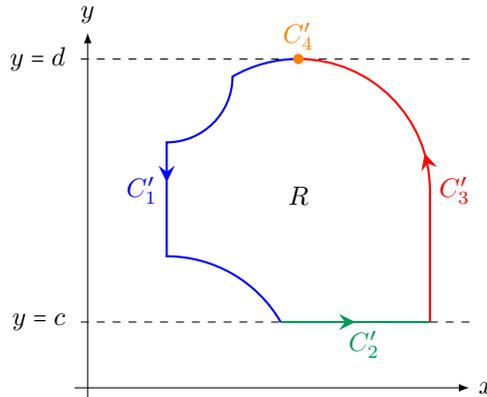(Note that these curves *may* be different from those chosen earlier.)

Figure 2: Realizing R as a Type II domain and labelling parts of its boundary ∂R.

A similar computation ultimately shows that

$$\oint_{\partial R} N\,dy = \sum_{k=1}^{4}\int_{C_k'} N\,dy = -\int_c^d \Big(N\big(h_1(t),t\big) - N\big(h_2(t),t\big)\Big)dt$$

$$= \int_c^d \Big(N\big(h_2(t),t\big) - N\big(h_1(t),t\big)\Big)dt$$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \frac{\partial N}{\partial x}\,dx\,dy$$

$$= \iint_R \frac{\partial N}{\partial x}\,dA. \tag{18}$$

Together, this gives

$$\oint_{\partial R} M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dA. \tag{19}$$

$\square$

Now that we have Green's Theorem for simple domains, we can use it to prove the case when R is a slightly more general region.

Consider the case where R is a larger (counter-clockwise oriented) domain consisting of smaller simple (counter-clockwise oriented) domains $R_1, R_2, \ldots, R_m$ that intersect along some unique boundary segments $L_{k,l}$ (see figure below). That is

- $R = \cup_{k=1}^m R_k$ where each R_k is simple,

- for $k \neq l$, $L_{k,l} = R_k \cap R_l = L_{l,k}$ is either empty or a curve (where we write $L_{k,l}$ to denote that it is oriented the same way as ∂R_k), and

- for $(k,l) \neq (k',l')$, $L_{k,l} \cap L_{k',l'}$ is some subset of the endpoints of $L_{k,l}$ and $L_{k',l'}$.

With this notation, we see that as oriented curves $L_{k,l} = -L_{l,k}$. Also,

$$\partial R_k = C_k + \sum_{l \neq k} L_{k,l} \tag{20}$$
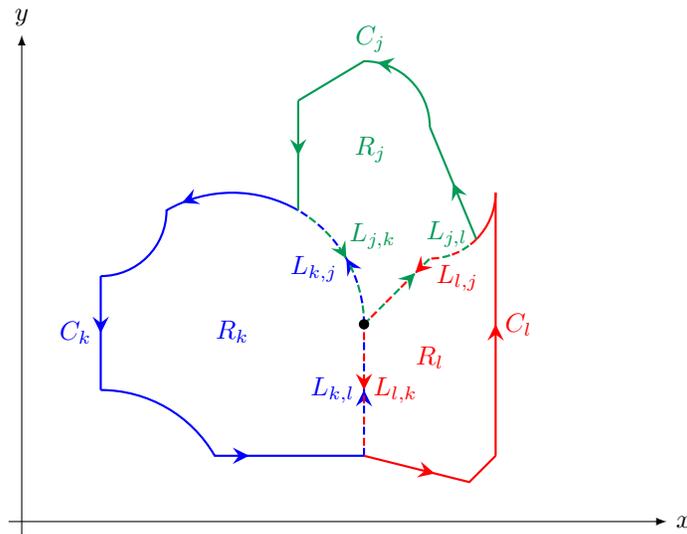
and

$$\partial R = \sum_{k=1}^m C_k \tag{21}$$

Figure 3: A more general domain R made up of simple domains.

We can compute (using Green's Theorem on each individual simple domain) that

$$
\begin{aligned}
\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA &= \sum_{k=1}^m \iint_{R_k} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \\
&= \sum_{k=1}^m \oint_{\partial R_k} M\, dx + N\, dy \\
&= \sum_{k=1}^m \oint_{C_k + \sum_{l \neq k} L_{k,l}} M\, dx + N\, dy \\
&= \sum_{k=1}^m \left(\int_{C_k} M\, dx + N\, dy + \sum_{l \neq k} \int_{L_{k,l}} M\, dx + N\, dy \right) \\
&= \sum_{k=1}^m \int_{C_k} M\, dx + N\, dy + \sum_{\substack{k,l=1 \\ k \neq l}}^n \int_{L_{k,l}} M\, dx + N\, dy \\
&= \oint_{\partial R} M\, dx + N\, dy,
\end{aligned}
\tag{22}
$$

where we have used that the second sum runs over all combinations of (k,l) with $k \neq l$ an the fact that $L_{k,l} = -L_{l,k}$.

This broader result works in almost all situations. To prove the general case, we need ideas from analysis, which is outside the scope of the course.

0.1 Differential Operators

We will now move on and discuss results involving some fairly ubiquitous differential operators in vector analysis.

Definition 2 (Divergence). Let

$$
\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}}.
\tag{23}
$$

We define the *divergence* $\operatorname{div} \vec{F}$ of $\vec{F}$ to be

$$
\operatorname{div} \vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}.
\tag{24}
$$

Note that the *divergence* of a vector field is a *scalar* quantity, not a vector.

Remark 3 (Flux Density). We can compute that

$$
\begin{aligned}
\operatorname{div} \vec{F} &= \lim_{\epsilon \to 0} \frac{1}{\operatorname{Area}\big(\overline{D}_\epsilon(x,y)\big)} \iint_{\overline{D}_\epsilon(x,y)} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA \\
&= \lim_{\epsilon \to 0} \frac{1}{\operatorname{Area}\big(\overline{D}_\epsilon(x,y)\big)} \oint_{\partial \overline{D}_\epsilon(x,y)} \vec{F} \cdot \hat{\mathbf{n}}\, ds,
\end{aligned}
\tag{25}
$$

where the final expression is a quantity called the *flux density*.

Remark 4. Recall that for a function $f(x,y)$ we defined the *gradient* operator $\vec{\nabla}$ by

$$
\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} = \left(\frac{\partial}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}\,\hat{\mathbf{j}} \right) f.
\tag{26}
$$

As such, it is convenient to denote

$$
\vec{\nabla} = \left(\frac{\partial}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}\,\hat{\mathbf{j}} \right)
\tag{27}
$$

and then

$$
\vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}\,\hat{\mathbf{j}} \right) \cdot \left(M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} \right) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \operatorname{div} \vec{F}.
\tag{28}
$$

Hence we also write

$$
\vec{\nabla} \cdot \vec{F} = \operatorname{div} \vec{F}.
\tag{29}
$$

Definition 5 (Rotation). Let

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}}. \tag{30}$$

We define the *rotation* rot $\vec{F}$ of $\vec{F}$ to be

$$\operatorname{rot}\vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \tag{31}$$

Like the *divergence*, the *rotation* is also a *scalar* quantity.

Remark 6 (Circulation Density). We have that

$$\operatorname{rot}\vec{F} = \lim_{\epsilon \to 0} \frac{1}{\operatorname{Area}\!\big(\overline{D}_\epsilon(x,y)\big)} \iint_{\overline{D}_\epsilon(x,y)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$$= \lim_{\epsilon \to 0} \frac{1}{\operatorname{Area}\!\big(\overline{D}_\epsilon(x,y)\big)} \oint_{\partial\overline{D}_\epsilon(x,y)} \vec{F} \cdot \hat{\mathbf{T}}\, ds, \tag{32}$$

where the final expression is a quantity called the *circulation density*.

By extending the notation

$$\vec{\nabla} = \frac{\partial}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial}{\partial y}\,\hat{\mathbf{j}} = \frac{\partial}{\partial z}\,\hat{\mathbf{k}}, \tag{33}$$

(note that we are still assuming $\vec{F}$ is a vector field in $\mathbb{R}^2$) we can write

$$\operatorname{rot}\vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{k}}. \tag{34}$$

Indeed, writing

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + 0\,\hat{\mathbf{k}} \tag{35}$$

we have

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ M & N \end{vmatrix} \hat{\mathbf{k}} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \hat{\mathbf{k}}, \tag{36}$$

and so

$$\operatorname{rot}\vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{\mathbf{k}} \tag{37}$$

as mentioned.

We call the vector $\vec{\nabla} \times \vec{F}$ the *curl* of $\vec{F}$, denoted curl $\vec{F}$.

Using this new notation, we can rewrite Green's Theorem more succinctly:

Theorem 7 (Green's Theorem (Differential Operator Form)). *Let $\Omega \subseteq \mathbb{R}^2$ be open and*

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} \tag{38}$$

be a C^1 vector field on Ω. If C is a piecewise smooth simple closed counter-clockwise oriented curve enclosing a region R lying entirely in Ω, then

$$[\,Normal\ Form\,] \qquad \oint_C \vec{F} \cdot \hat{\mathbf{n}}\, ds \iint_R \operatorname{div}\vec{F}\, dA \tag{39}$$

$$[\,Tangential\ Form\,] \qquad \oint_C \vec{F} \cdot \hat{\mathbf{T}}\, ds = \iint_R \operatorname{rot}\vec{F}\, dA. \tag{40}$$

We also have a new characterization of conservative vector fields:

Theorem 8. *Suppose $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) is open, (path) connected and simply connected. Let $\vec{F}$ be a C^1 vector field on Ω. Then*

$$\vec{F} \text{ is conservative on } \Omega \iff \operatorname{curl}\vec{F} = 0. \tag{41}$$

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(End of Lecture 15 – Oct 30)

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