MATH2020A Lecture 14 Notes

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Last time, we saw that if $\vec{F}$ is a C^1 vector field on an open, (path) connected domain $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3), then

$$\vec{F} = \vec{\nabla} f \text{ for some function } f \iff \vec{F} \text{ is conservative.} \tag{1}$$

We have seen that for a C^1 vector field

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}} \tag{2}$$

on an open, (path) connected domain $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) that

$$\vec{F} \text{ conservative} \implies M,\ N, \text{ and } L \text{ satisfy the system of PDE's } (\star). \tag{3}$$

Where the system $(\star)$ is

$$(\star) = \begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}, \\ \dfrac{\partial N}{\partial z} = \dfrac{\partial L}{\partial y}, \\ \dfrac{\partial L}{\partial x} = \dfrac{\partial M}{\partial z}, \end{cases} \tag{4}$$

when $n = 3$ and

$$(\star) = \begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}, \end{cases} \tag{5}$$

when $n = 2$.

Is the converse true? No, at least not in general. We need an extra condition on the domain Ω ((path) connectedness is not enough).

Example 1. Consider the vector field

$$\vec{F} = \frac{-y}{x^2 + y^2}\,\hat{\mathbf{i}} + \frac{x}{x^2 + y^2}\,\hat{\mathbf{j}} \tag{6}$$

and the domains

$$\Omega_1 = \mathbb{R}^2 \backslash \{(x,0) \in \mathbb{R}^2 \mid x \leq 0\} \text{ and } \Omega^2 = \mathbb{R}^2 \backslash \{(0,0)\}. \tag{7}$$

(Note that Ω_1 does not include the negative x-axis and the origin while Ω_2 only does not include the origin.)

In polar coordinates,

$$\vec{F} = -\frac{\sin\theta}{r}\,\hat{\mathbf{i}} + \frac{\cos\theta}{r}\,\hat{\mathbf{j}} \tag{8}$$

We see that $\vec{F}$ rotates around the origin in a counter-clockwise manner. Also

$$\|\vec{F}\| = \frac{1}{r} \to 0 \text{ as } r \to \infty, \tag{9}$$

while

$$\|\vec{F}\| = \frac{1}{r} \to \infty \text{ as } r \to 0. \tag{10}$$

As such, we *cannot* extend $\vec{F}$ to a C^1 vector field on *all* of $\mathbb{R}^2$.

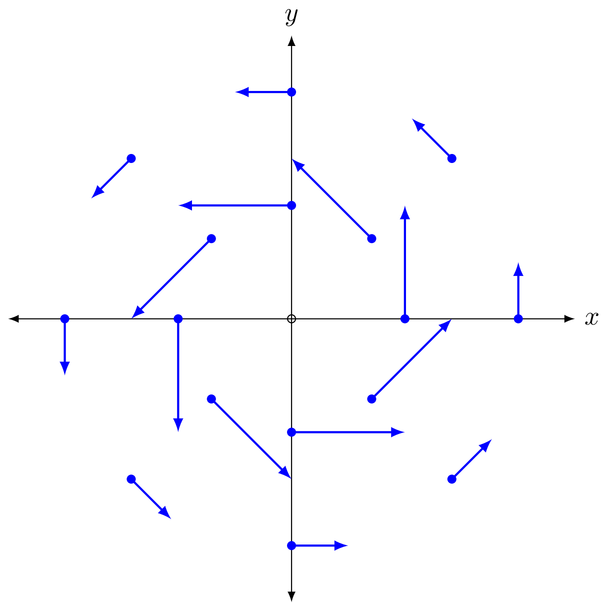


Figure 1: The vector field $\vec{F}(r,\theta) = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$.

With the exception of the origin, $\vec{F}$ is C^1 and in particular, it is C^1 on both Ω_1 and Ω_2.

Is $\vec{F}$ conservative on Ω_1? What about on Ω_2?

Solution. We can represent Ω_1 in polar coordinates as

$$\Omega_1 = \{(r,\theta) \in \mathbb{R}^2 \mid r > 0, -\pi < \theta < \pi\}. \tag{11}$$

The inequalities are strict since we *do not* contain the negative x-axis.

The function $f(r,\theta) = \theta$ (where θ is restricted to $(-\pi,\pi)$) is smooth on Ω_1. We can show that

$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial \theta}{\partial x} = -\dfrac{\sin\theta}{r}, \\ \dfrac{\partial f}{\partial y} = \dfrac{\partial \theta}{\partial y} = \dfrac{\cos\theta}{r}. \end{cases} \tag{12}$$

[**Exercise** : Check this] As such, we have

$$\vec{F} = -\frac{\sin\theta}{r}\,\hat{\mathbf{i}} + \frac{\cos\theta}{r}\,\hat{\mathbf{j}} = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} = \vec{\nabla} f, \tag{13}$$

hence $\vec{F}$ is conservative on Ω_1.

On the other hand, we *cannot* extend the angle function $f(r,\theta) = \theta$ *continuously* to all of Ω_2. This is because the angle function "jumps" from $-\pi$ to π along the negative x-axis. This means that the function $f = \theta$ *cannot* be a scalar potential.

To show that $\vec{F}$ is *not* conservative on Ω_2, we consider a closed curve

$$C : \vec{r}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}, \qquad t \in [-\pi,\pi]. \tag{14}$$

(This is the unit circle, which is contained in Ω_2 but not Ω_1.)

Then we have

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{-\pi}^{\pi}\left(-\frac{\sin\theta}{r}\,\hat{\mathbf{i}} + \frac{\cos\theta}{r}\,\hat{\mathbf{j}}\right)\cdot\left((\cos t)'\,\hat{\mathbf{i}} + (\sin t)'\,\hat{\mathbf{j}}\right)dt$$
$$= \int_{-\pi}^{\pi}\left((-\sin\theta)^2 + \cos^2\theta\right)dt \qquad [r = 1 \text{ along this curve}]$$
$$= \int_{-\pi}^{\pi} dt = 2\pi \neq 0.$$

By a previous result, $\vec{F}$ is *not* conservative. □

The main difference between the sets Ω_1 and Ω_2 from the previous example is the following property:

Definition 2 (Simply Connected)**.** A subset $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) is called *simply connected* if *every* closed curve in Ω can be *contracted (deformed continuously)* to a point in Ω (without leaving Ω).

In some sense, one can think of a simply connected set as one without any "holes".

Example 3. The set

$$\Omega_1 = \mathbb{R}^2 \backslash \{(x,0) \in \mathbb{R}^2 \mid x \leq 0\} \tag{15}$$

is simply connected, while

$$\Omega_2 = \mathbb{R}^2 \backslash \{(0,0)\} \tag{16}$$

is not.

We can see that Ω_2 is not simply connected since the curve C given by the unit circle cannot be deformed into a point without passing through the origin. This problem does not occur for Ω_1 since curves need to "skirt around" the origin to stay in Ω_1.

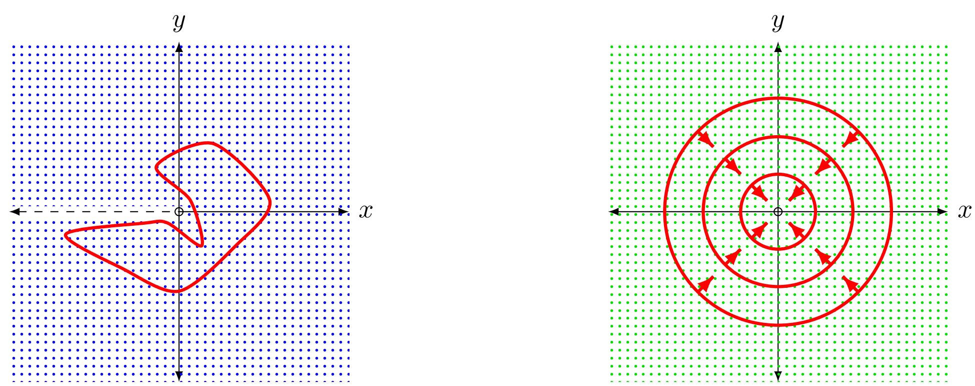


Figure 2: The set Ω_1 is simply connected but the set Ω_2 is not.

Example 4. The unit sphere

$$S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3 \tag{17}$$

is simply connected. Any curve on S^2 can be "pushed" towards one of the poles (sort of like a rubber band on the surface of a ball).

The torus

$$T^2 = S^1 \times S^1 \subseteq \mathbb{R}^3 \tag{18}$$

is *not* simply connected. There are two distinct types of closed curves on it that cannot be contracted to a point.

Remark 5. Simply connectedness is a *global* condition that guarantees that satisfying the system of PDE's $(\star)$ implies that $\vec{F}$ is conservative.

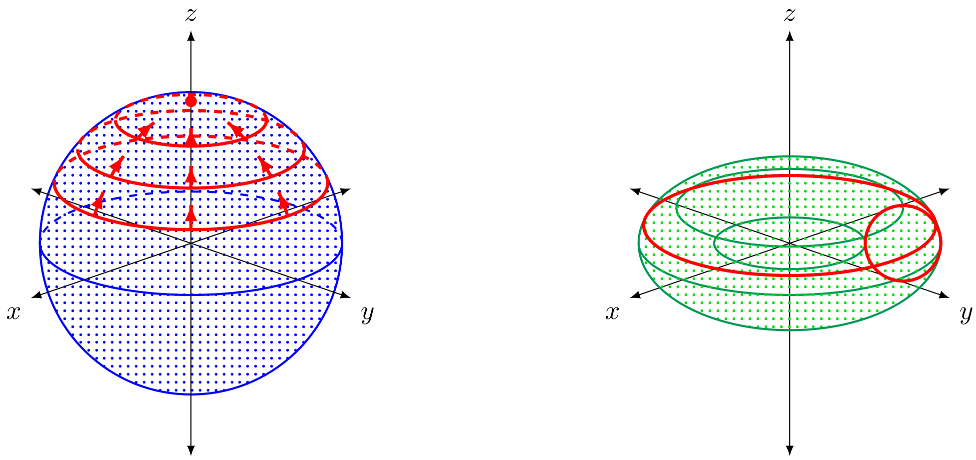


Figure 3: The sphere S^2 is simply connected but the torus T^2 is not.

Theorem 6. *Suppose* $\Omega \subseteq \mathbb{R}^n$ *(*$n = 2$ *or* 3*) is* open, (path) connected *and* simply connected. *Let* $\vec{F}$ *be a* C^1 *vector field on* Ω*. Then*

$$\vec{F} \text{ is conservative on } \Omega \iff \text{ the components of } F \text{ satisfy the system of PDE's } (\star). \tag{19}$$

Proof. Postponed. ☐

Example 7 (Finding the Potential of a Vector Field)**.** Let $\Omega = \mathbb{R}^3$ (which is open, (path) connected and simply connected) and

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}} = (y + e^z)\,\hat{\mathbf{i}} + (x+1)\,\hat{\mathbf{j}} + (1 + xe^z)\,\hat{\mathbf{k}}. \tag{20}$$

Find the *scalar potential function* f of $\vec{F}$, *i.e.* $\vec{\nabla} f = \vec{F}$.

Solution. We want to solve the system

$$\frac{\partial f}{\partial x} = M = y + e^z, \quad \frac{\partial f}{\partial y} = N = x + 1, \quad \frac{\partial f}{\partial z} = L = 1 + xe^z. \tag{21}$$

We first verify that the system $(\star)$ is satisfied.

$$\begin{cases} \dfrac{\partial M}{\partial y} = 1 = \dfrac{\partial N}{\partial x}, \\ \dfrac{\partial N}{\partial z} = 0 = \dfrac{\partial L}{\partial y}, \\ \dfrac{\partial L}{\partial x} = e^z = \dfrac{\partial M}{\partial z}. \end{cases} \tag{22}$$

Thus, by the previous theorem, a scalar potential function f exists.

To find f explicitly, we first use that

$$\frac{\partial f}{\partial x} = M = y + e^z. \tag{23}$$

Integrating this, we find that

$$f(x,y,z) = \int (y + e^z)\,dx = x(y + e^z) + C \tag{24}$$

where C is a constant in x. This constant *could* depend on y and z and so we will write $C = g(y,z)$ to reflect this possibility. That is

$$f = x(y + e^z) + g(y,z). \tag{25}$$

Let us take the partial derivative of this with respect to y to get

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}. \tag{26}$$

By our system, this should be equal to $N = x + 1$ and so by matching, we get that

$$\frac{\partial g}{\partial y} = 1 \tag{27}$$

and so by a similar reasoning,

$$g = \int 1\,dy = y + h(z) \tag{28}$$

where h is a function that depends on z (and not x or y). This gives that

$$f(x,y,z) = x(y + e^z) + y + h(z). \tag{29}$$

Taking the partial derivative of f with respect to z, we finally see that

$$\frac{\partial f}{\partial z} = xe^z + \frac{\partial h}{\partial z}. \tag{30}$$

Again, by our system, this should be equal to $L = 1 + xe^z$ and so we conclude that

$$\frac{\partial h}{\partial z} = 1 \implies h(z) = z + C, \tag{31}$$

where C is a constant.

We hence see that

$$f(x,y,z) = x(y + e^z) + y + z + C \tag{32}$$

are the scalar potentials for $\vec{F}$. □

Remark 8. Recall that scalar potentials are unique up to addition of a constant, which is reflected in the example above.

Remark 9. As previously mentioned, finding a scalar potential f is equivalent to finding a function f such that

$$df = M\,dx + N\,dy + L\,dz. \tag{33}$$

In this case, the *differential* 1*-form* $M\,dx + N\,dy + L\,dz$ is said to be *exact*.

In order to prove Theorem 6 in the $\mathbb{R}^2$ case, we need *Green's Theorem* (for $\mathbb{R}^3$ we need *Stokes' Theorem*).

0.1 Green's Theorem

Theorem 10 (Green's Theorem)**.** *Let $\Omega \subseteq \mathbb{R}^2$ be open and*

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} \tag{34}$$

be a C^1 vector field on Ω. If C is a piecewise smooth simple closed counter-clockwise oriented curve enclosing a region R lying entirely in Ω, then

$$[\textit{Normal Form}] \qquad \oint_C \vec{F}\cdot\hat{\mathbf{n}}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy, \tag{35}$$

$$[\textit{Tangential Form}] \qquad \oint_C \vec{F}\cdot\hat{\mathbf{T}}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy. \tag{36}$$

Remark 11. The two forms are equivalent.

Remark 12. Reconsidering Example 3, we see that Green's Theorem applies for any curve in Ω_1 since the enclosed region R will lie entirely in Ω_1.

On the other hand, Green's Theorem will *not* apply to any curve in Ω_2 containing the origin.

Example 13. Verify both forms of Green's Theorem for the vector field

$$\vec{F}(x,y) = (x - y)\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} \tag{37}$$

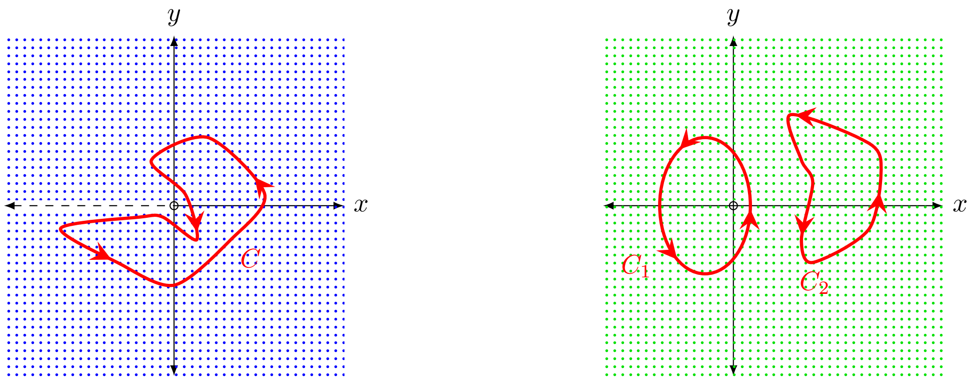


Figure 4: Green's Theorem will apply to the curves $C \subseteq \Omega_1$ and $C_2 \subseteq \Omega_2$ since the regions they enclose are entirely in the respective sets. Green's Theorem will *not* apply for the curve $C_1 \subseteq \Omega_2$.

on $\Omega = \mathbb{R}^2$ with

$$C\colon \vec{r}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}, \qquad t \in [0, 2\pi]. \tag{38}$$

Solution. We first notice that the region R bounded by C is the (open) unit disc

$$R = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}, \tag{39}$$

which is contained in Ω and so Green's Theorem should apply. (We also write $\partial R = C$ where ∂R denotes the *boundary of* R.)

In this case,

$$\begin{cases} M = x - y, \\ N = x, \end{cases} \qquad \Longrightarrow \begin{cases} \dfrac{\partial M}{\partial x} = 1, \\ \dfrac{\partial M}{\partial y} = -1, \\ \dfrac{\partial N}{\partial x} = 1, \\ \dfrac{\partial N}{\partial y} = 0. \end{cases} \tag{40}$$

Along the curve C (with counter-clockwise parameterization $\vec{r}$), we also have

$$x = \cos t, \quad y = \sin t, \qquad t \in [0, 2\pi]. \tag{41}$$

For the normal form of Green's Theorem, we note that

$$\begin{aligned} \text{LHS} &= \oint_C M\,dy - N\,dx \\ &= \int_0^{2\pi} (\cos t - \sin t)\,d(\sin t) - \cos t\,d(\cos t) \\ &= \int_0^{2\pi} \Big((\cos t - \sin t)(\cos t) + \cos t \sin t\Big)\,dt \\ &= \int_0^{2\pi} \cos^2 t\,dt \\ &= \int_0^{2\pi} \frac{\cos(2t) + 1}{2}\,dt \\ &= \left(\frac{\sin(2t)}{4} + \frac{t}{2}\right)\Bigg|_{t=0}^{t=2\pi} = \pi. \end{aligned} \tag{42}$$

We also have

$$\begin{aligned}\text{RHS} &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \\ &= \iint_R (1+0)\,dx\,dy \\ &= \pi. \end{aligned} \tag{43}$$

We can also check the tangential form of Green's Theorem:

$$\begin{aligned}\text{LHS} &= \oint_C M\,dx + N\,dy \\ &= \int_0^{2\pi} (\cos t - \sin t)\,d(\cos t) + \cos t\,d(\sin t) \\ &= \int_0^{2\pi} \Big(-(\cos t - \sin t)(\sin t) + \cos^2 t\Big)\,dt \\ &= \int_0^{2\pi} \Big(1 - \cos t \sin t\Big)\,dt \\ &= 2\pi. \end{aligned} \tag{44}$$

Also,

$$\begin{aligned}\text{RHS} &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \\ &= \iint_R \big(1-(-1)\big)\,dx\,dy \\ &= 2\pi. \end{aligned}$$

□

Remark 14. Even though both forms of Green's Theorem are equivalent, the actual values involved may differ.

Recall domains $D \subseteq \mathbb{R}^2$ of special type:

- (Type I) $D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$, where g_1 and g_2 are *continuous* functions on $[a,b]$ with $g_1(x) \le g_2(x)$ for $x \in [a,b]$,
- (Type II) $D = \{(x,y) \in \mathbb{R}^2 \mid h_1(y) \le x \le h_2(y), c \le y \le d\}$, where h_1 and h_2 are *continuous* functions on $[c,d]$ with $h_1(x) \le h_2(x)$ for $y \in [c,d]$.

If D is of both Type I and Type II, it is said to be *simple*.

Example 15. A rectangle $R = [a,b] \times [c,d]$ is simple.

Example 16. The region R in the first quadrant bounded by the circles

$$C_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \tag{45}$$

and

$$C_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\} \tag{46}$$

is simple.

Proposition 17 (Characterization of Simple Domains)**.** *Let D be a domain and suppose that ∂D is piecewise smooth. If for all $a \in \mathbb{R}$ we have*

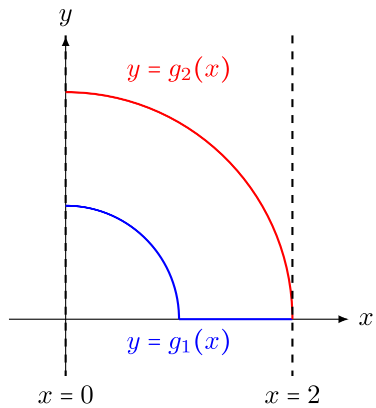
$$\#\Big\{\partial D \cap \{x = a\}\Big\} \le 2 \tag{47}$$

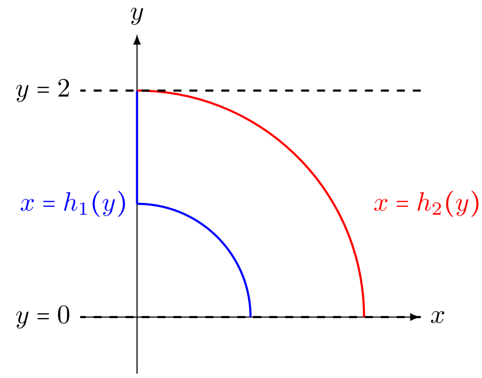
and

$$\#\Big\{\partial D \cap \{y = a\}\Big\} \le 2 \tag{48}$$

then D is simple.

Proof. Omitted. □





Proof of Green's Theorem for Simple Domains. Let R be a simple domain, as such R is Type I and we can write

$$R = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}. \tag{49}$$

Denote the four components of the boundary ∂R of R by the curves C_1, C_2, C_3, and C_4 as in the figure. (Note that it is possible for C_2 or C_4 to be a point.) We then have

$$\partial R = C_1 + C_2 + C_3 + C_4 \tag{50}$$

as oriented curves.

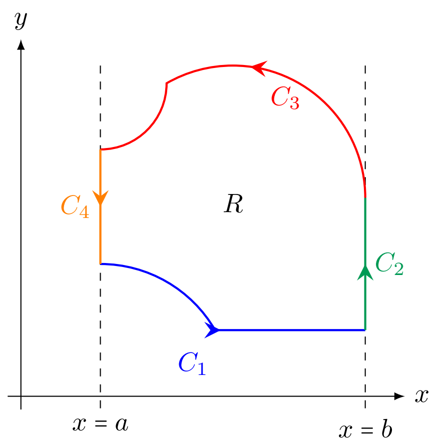


Figure 5: Realizing R as a Type I domain and labelling parts of its boundary ∂R.

We can parameterize

$$C_1 = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b, y = g_1(x)\} \tag{51}$$

by

$$\vec{r}_1(t) = \big(t, g_1(t)\big), \qquad t \in [a,b]. \tag{52}$$

Likewise, we can parameterize $-C_3$ by

$$\vec{r}_3(t) = \big(t, g_2(t)\big), \qquad t \in [a,b]. \tag{53}$$

Note that we have to ensure that the parameterizations give the proper orientations of the curves.

Hence

$$\int_{C_1} M\,dx = \int_a^b M\big(t, g_1(t)\big)\,dt \text{ and } \int_{-C_3} M\,dx = \int_a^b M\big(t, g_2(t)\big)\,dt. \tag{54}$$

For the vertical segments, we have

$$C_2 = \{(x,y) \in \mathbb{R}^2 \mid x = b, g_1(b) \le y \le g_2(b)\} \tag{55}$$

and so we can parameterize is by

$$\vec{r}_2(t) = (b,t), \qquad t \in [g_1(b), g_2(b)]. \tag{56}$$

Similarly, we can parameterize $-C_4$ by

$$\vec{r}_4(t) = (a,t), \qquad t \in [g_1(a), g_2(a)]. \tag{57}$$

Hence

$$\int_{C_2} M\,dx = \int_{g_1(b)}^{g_2(b)} M(b,t)\,db = 0 \text{ and } \int_{-C_4} M\,dx = \int_{g_1(a)}^{g_2(a)} M(a,t)\,da = 0. \tag{58}$$

Putting everything together, we get

$$\oint_{\partial R} M\,dx = \sum_{k=1}^{4} \int_{C_k} M\,dx = \int_a^b \Big(M\big(t, g_1(t)\big) - M\big(t, g_2(t)\big) \Big)\,dt. \tag{59}$$

On the other hand, using Fubini's Theorem, we have

$$\begin{aligned} \iint_R \left(-\frac{\partial M}{\partial y} \right) dA &= \int_a^b \int_{g_1(x)}^{g_2(x)} \left(-\frac{\partial M}{\partial y} \right) dy\,dx \\ &= \int_a^b -\Big(M\big(x, g_2(x)\big) - M\big(x, g_1(x)\big) \Big)\,dx \\ &= \oint_{\partial R} M\,dx. \end{aligned} \tag{60}$$

(End of Lecture 14 – Oct 27) □