MATH2020A Lecture 13 Notes

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Last time, we considered the notion of a conservative vector field:

Definition 1 (Conservative Vector Field)**.** Let $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) be open. A vector field $\vec{F}$ on Ω is *conservative* if the integral

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds = \int_C \vec{F} \cdot d\vec{r} \tag{1}$$

along an *oriented* curve C in Ω *depends only* on the *start and end points* of C.

We also looked at the Fundamental Theorem of Path Integrals.

Theorem 2 (Fundamental Theorem of Path Integrals)**.** *Let f be a C^1 function on an open set $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) and let $\vec{F} = \vec{\nabla} f$ be the* gradient vector field *of f. Then, for any piecewise smooth oriented curve C in Ω with start point A and end point B, we have*

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds = f(B) - f(A). \tag{2}$$

Using this, we saw that every gradient vector field $\vec{F} = \vec{\nabla} f$ is conservative. We then considered the question of the converse:

Question 3. Is every conservative vector field a gradient vector field?

This led us to a partial converse that required the domain Ω that we are working in to be open and (path) connected.

Theorem 4 (Conservative Potentials on Connected Domains)**.** *Let $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) be open and* (path) connected *and let $\vec{F}$ be a* continuous *vector field on Ω. Then the following are equivalent:*

i) there exists a C^1 function $f: \Omega \to \mathbb{R}$ such that

$$\vec{F} = \vec{\nabla} f, \tag{3}$$

ii) for any closed curve *C on Ω, we have*

$$\oint_C \vec{F} \cdot d\vec{r} = 0, \tag{4}$$

iii) the vector field $\vec{F}$ is conservative.

Proof. $[i) \implies ii)]$

If f is C^1 and $\vec{F} = \vec{\nabla} f$ and

$$\vec{r}: [a, b] \to \Omega \tag{5}$$

parameterizes the closed curve C, then

$$\vec{r}(a) = \vec{r}(b) = A. \tag{6}$$

The Fundamental Theorem of Path Integrals then tells us that

$$\oint_C \vec{F} \cdot d\vec{r} = f\big(\vec{r}(b)\big) - f\big(\vec{r}(a)\big) = f(A) - f(A) = 0. \tag{7}$$

[$ii) \implies iii)$]

Suppose C_1 and C_2 are oriented curves with start point A and end point B. Then $C_1 - C_2$ is an oriented closed curve with start and end point A.

Figure 1: Creating a closed curve from two oriented curves with the same start and end points.

By ii), we have

$$\begin{aligned} 0 &= \oint_{C_1 - C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r} \\ &= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}. \end{aligned} \tag{8}$$

Since C_1 and C_2 were arbitrary curves, we see that $\vec{F}$ is conservative.

[$iii) \implies i)$]

This requires us to solve a system of partial differential equations (PDE's).

Let us work with the $n = 2$ case for now (the other case will be similar).

Suppose $\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}}$ is conservative. Our goal is to find a C^1 function f such that

$$\frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} = \vec{\nabla} f = \vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}}. \tag{9}$$

This is equivalent to finding $f \in C^1$ such that we solve the system of PDE's

$$\begin{cases} \frac{\partial f}{\partial x} = M, \\ \frac{\partial f}{\partial y} = N, \end{cases} \tag{10}$$

(which is also equivalent to $df = M\,dx + N\,dy$.)

Fix some point $A \in \Omega$. Then for any point B define

$$f(B) = \int_A^B \vec{F} \cdot \hat{\mathbf{T}}\,ds \tag{11}$$

where the integral is the *common value* of $\int_C \vec{F} \cdot \hat{\mathbf{T}}\,ds$ for *any* curve C oriented from A to B.

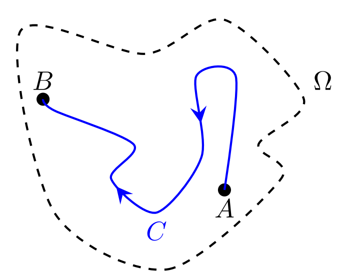

Figure 2: Defining the function f using path integrals over arbitrary curves C.

Since $\vec{F}$ is conservative, this is well-defined. We also used the *path connected* assumption to ensure we can even define $f(B)$ in the first place.

We claim that $\vec{\nabla} f = \vec{F}$. To show this, we notice that

$$\frac{\partial f}{\partial x}(B) = \lim_{\epsilon \to 0} \frac{f(B + \epsilon\,\hat{\mathbf{i}}) - f(B)}{\epsilon}. \tag{12}$$

Let L be the horizontal line segment from B to $B + \epsilon\,\hat{\mathbf{i}}$ with $|\epsilon|$ sufficiently small that $B + \epsilon\,\hat{\mathbf{i}} \in \Omega$ (this is possible since Ω is open).

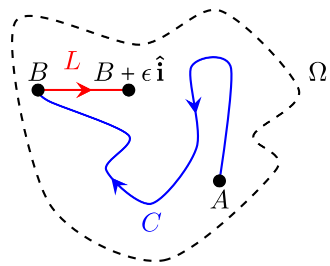

Figure 3: Finding the derivative of f with respect to x.

Then we have

$$\begin{aligned} f(B + \epsilon\,\hat{\mathbf{i}}) &= \int_A^{B+\epsilon\,\hat{\mathbf{i}}} \vec{F} \cdot \hat{\mathbf{T}}\, ds = \int_{C+L} \vec{F} \cdot \hat{\mathbf{T}}\, ds \\ &= \int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds + \int_L \vec{F} \cdot \hat{\mathbf{T}}\, ds \\ &= f(B) + \int_L \vec{F} \cdot \hat{\mathbf{T}}\, ds \end{aligned} \tag{13}$$

and so

$$\frac{f(B + \epsilon\,\hat{\mathbf{i}}) - f(B)}{\epsilon} = \frac{1}{\epsilon} \int_L \vec{F} \cdot \hat{\mathbf{T}}\, ds = \frac{1}{\epsilon} \int_L \vec{F} \cdot d\vec{r}. \tag{14}$$

Since we can parameterize the curve L by

$$(x + t, y) \qquad t \in [0, \epsilon] \tag{15}$$

where $B = (x, y)$, we have

$$\begin{aligned} \frac{f(B + \epsilon\,\hat{\mathbf{i}}) - f(B)}{\epsilon} &= \frac{1}{\epsilon} \int_0^{\epsilon} \left(M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} \right) \cdot \left((x+t)'\,\hat{\mathbf{i}} + y'\,\hat{\mathbf{j}} \right) dt \\ &= \frac{1}{\epsilon} \int_0^{\epsilon} M(x + t, y)\, dt \to M(x, y) \text{ as } \epsilon \to 0 \end{aligned} \tag{16}$$

by the Mean Value Theorem and the fact that M is continuous (or by the Fundamental Theorem of Calculus).

It follows that

$$\frac{\partial f}{\partial x}(B) = \frac{\partial f}{\partial x}(x, y) = M(x, y). \tag{17}$$

A similar consideration using a vertical (straight) line segment gives that

$$\frac{\partial f}{\partial y}(B) = \frac{\partial f}{\partial y}(x, y) = N(x, y). \tag{18}$$

As such, $\vec{\nabla} f = \vec{F}$.

Since $\vec{F}$ is continuous, we have

$$\frac{\partial f}{\partial x} = M \text{ and } \frac{\partial f}{\partial y} = N \tag{19}$$

are continuous, which imply that f is C^1. □

Corollary 5. *Let $\vec{F}$ be* conservative *and C^1 on an open, (path) connected domain $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3).*

If $n = 3$ and

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}}, \tag{20}$$

then

$$(\star) = \begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}, \\ \dfrac{\partial N}{\partial z} = \dfrac{\partial L}{\partial y}, \\ \dfrac{\partial L}{\partial x} = \dfrac{\partial M}{\partial z}. \end{cases} \tag{21}$$

If instead, $n = 2$ and

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}}, \tag{22}$$

then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{23}$$

Proof. If $\vec{F}$ is conservative, then by the previous theorem, we have a scalar potential f such that

$$\vec{F} = \vec{\nabla} f. \tag{24}$$

Since $\vec{F} \in C^1$, we must have that $f \in C^2$.

By Clairaut's Theorem, we have

$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial N}{\partial x}, \\ \dfrac{\partial N}{\partial z} = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial z}\right) = \dfrac{\partial L}{\partial y}, \\ \dfrac{\partial L}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial z}\right) = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial M}{\partial z}. \end{cases} \tag{25}$$

The 2-dimensional case is similar. □

Example 6 (A Non-Conservative Vector Field)**.** Show that

$$\vec{F}(x, y) = \hat{\mathbf{i}} + x\,\hat{\mathbf{j}} \tag{26}$$

is *not* conservative in $\mathbb{R}^2$.

Proof. We have that $\vec{F}$ is smooth and that $M = 1$ and $N = x$. Computing, we have

$$\frac{\partial M}{\partial y} = 0 \text{ and } \frac{\partial N}{\partial x} = 1. \tag{27}$$

Since these are different, we conclude that $\vec{F}$ is *not* conservative. □

(End of Lecture 13 – Oct 23)