MATH2020A Lecture 11 Notes

Caleb Suan

Last time, we looked at another way to integrate along a curve. Instead of a scalar function f, we considered a vector field $\vec{F}$ and defined the integral along an (oriented) curve C with parameterization $\vec{r}(t)$ as

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds = \int_C \vec{F} \cdot d\vec{r}. \tag{1}$$

0.0.1 Physical Quantities

Certain physical quantities can be described using vector analysis and path integrals.

Work

Definition 1 (Work Done by a Force)**.** Let $\vec{F}(x, y)$ be a vector field (representing a force) and let C be an oriented curve, then the *work done by the field $\vec{F}$ along C* is given by

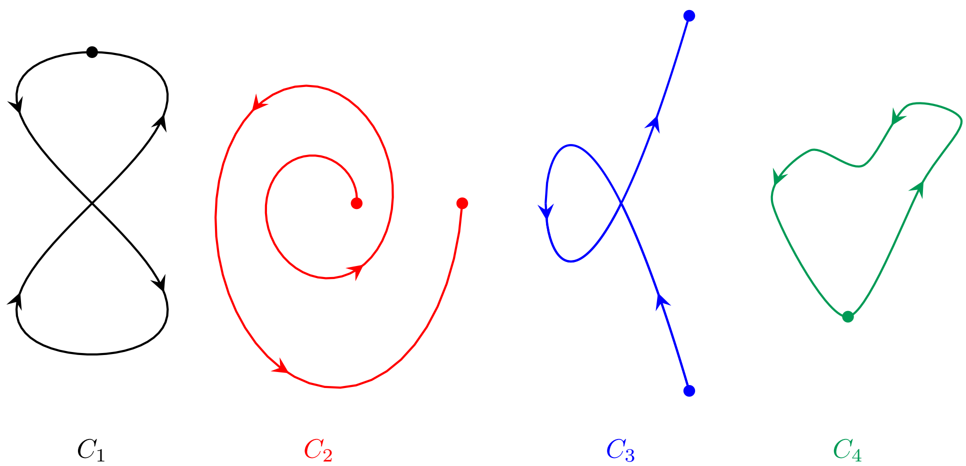
$$W = \int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds. \tag{2}$$

Flow

Definition 2 (Simple and Closed Curves)**.** A curve C is called:

- *simple* if it *does not* intersect itself except possibly at the end points,
- *closed* if its starts and ends at the same point.

Example 3. Some examples of curves:



We have the following properties:

- the curve C_1 is *closed*, but *not simple*;
- the curve C_2 is *simple*, but not closed;

- the curve C_3 is *not simple* and *not closed*;
- the curve C_4 is *simple* and *closed*.

Definition 4 (Flow Along a Curve)**.** Let $\vec{F}(x, y)$ be a vector field (usually the velocity field of a fluid) and C be an oriented curve. The *flow of the field $\vec{F}$ along C* is given by

$$\text{Flow} = \int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds. \tag{3}$$

If the curve C is *closed*, the flow is also called the *circulation*.

Remark 5. This is the same quantity as work, just used with a different context.

Flux We have the following result about simple closed plane curves.

Theorem 6 (Jordan Curve Theorem)**.** *Every* Jordan curve *(simple closed plane curve) divides the plane into a (bounded)* interior *region and an (unbounded)* exterior *region.*

Proof. Omitted. □

While this result seems obvious and intuitive, proving it turns out to be quite difficult and beyond the scope of this course. We will, however, be making use of the Jordan Curve Theorem to define certain objects.

Let C be a simple, closed, oriented plane curve (so $C \subseteq \mathbb{R}^2$) with parameterization

$$\vec{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}. \tag{4}$$

We have two choices of *unit normal vectors*. The Jordan Curve Theorem allows us to consistently choose our *unit normal* $\hat{\mathbf{n}}$ such that it that points "outward" towards the (unbounded) *exterior* region.

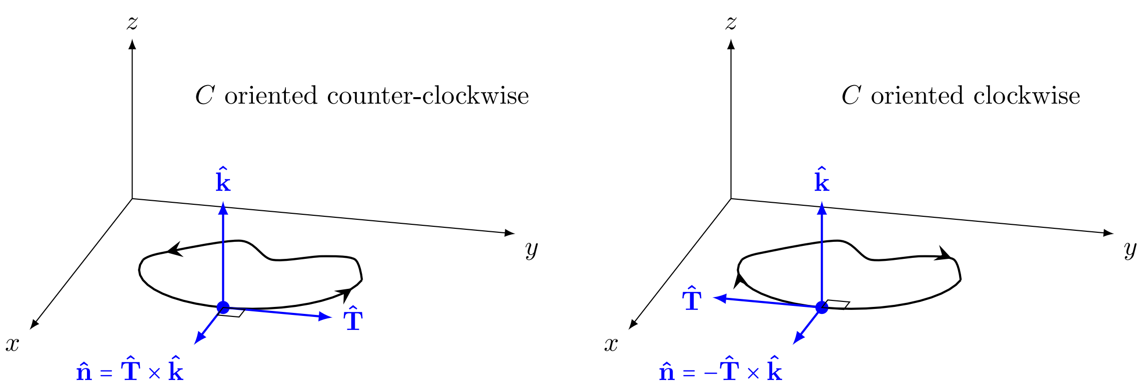


Figure 1: The outward unit normal vector $\hat{\mathbf{n}}$ of a simple closed plane curve.

We can compute the *unit normal vector* $\hat{\mathbf{n}}$ using the cross product $\times$, with a sign depending on whether the curve is oriented counter-clockwise or clockwise.

Recall that the unit tangent vector $\hat{\mathbf{T}}$ is given by

$$\hat{\mathbf{T}} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{x'(t)\,\hat{\mathbf{i}} + y'(t)\,\hat{\mathbf{j}}}{\|\vec{r}'(t)\|} \tag{5}$$

(Using an *arc-length parameterization* we have

$$\hat{\mathbf{T}} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\,\hat{\mathbf{i}} + \frac{dy}{ds}\,\hat{\mathbf{j}} \tag{6}$$

instead.)

If C is oriented counter-clockwise, then

$$\hat{\mathbf{n}} = \hat{\mathbf{T}} \times \hat{\mathbf{k}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{x'}{\|\vec{r}'\|} & \frac{y'}{\|\vec{r}'\|} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{y'(t)\,\hat{\mathbf{i}} - x'(t)\,\hat{\mathbf{j}}}{\|\vec{r}'(t)\|}. \tag{7}$$

(Using an *arc-length parameterization* we have

$$\hat{\mathbf{n}} = \frac{dy}{ds}\,\hat{\mathbf{i}} - \frac{dx}{ds}\,\hat{\mathbf{j}} \tag{8}$$

instead.)

If C is oriented clockwise, then the formulae become

$$\hat{\mathbf{n}} = \frac{-y'(t)\,\hat{\mathbf{i}} + x'(t)\,\hat{\mathbf{j}}}{\|\vec{r}'(t)\|} \tag{9}$$

(or

$$\hat{\mathbf{n}} = -\frac{dy}{ds}\,\hat{\mathbf{i}} + \frac{dx}{ds}\,\hat{\mathbf{j}} \tag{10}$$

with an *arc-length parameterization.*)

Definition 7 (Flux through a Simple Closed (Plane) Curve)**.** Let $\vec{F}(x,y)$ be a vector field (usually the velocity field of a fluid) and C be a simple closed oriented plane curve. The *flux of $\vec{F}$ through C* is the quantity

$$\text{Flux} = \oint_C \vec{F} \cdot \hat{\mathbf{n}}\,ds. \tag{11}$$

Remark 8. The notation

$$\oint_C \vec{F} \cdot \hat{\mathbf{n}}\,ds \tag{12}$$

is used to denote that we are integrating over a *closed* curve.

To emphasize the orientation, we sometimes use the notation

$$\ointctrclockwise_C \vec{F} \cdot \hat{\mathbf{n}}\,ds \tag{13}$$

for an integral over a *counter-clockwise oriented closed curve* and

$$\ointclockwise_C \vec{F} \cdot \hat{\mathbf{n}}\,ds \tag{14}$$

for an integral over a *clockwise oriented closed curve.*

Sometimes, if no direction is specified, one must determine the orientation by context. In most cases, the convention is that integrals orient the curve counter-clockwise.

Remark 9. The *flux* of a field $\vec{F}$ through a curve C describes "how much of the field $\vec{F}$ passes through C".

As of now, it is not geometrically clear how this dot product measures the described quantity. This will be a consequence of Green's Theorem (to come later).

In coordinates, if

$$\vec{F}(x,y) = M(x,y)\,\hat{\mathbf{i}} + N(x,y)\,\hat{\mathbf{j}} \tag{15}$$

and C is a simple closed plane curve parameterized anti-clockwise by

$$\vec{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}, \tag{16}$$

then

$$\begin{aligned}\text{Flux} &= \oint_C (M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}}) \cdot \left(\frac{dy}{ds}\,\hat{\mathbf{i}} - \frac{dx}{ds}\,\hat{\mathbf{j}}\right) ds \\ &= \oint_C M\,dy - N\,dx.\end{aligned}$$

Example 10. Let

$$\vec{F}(x,y) = (x-y)\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}} \tag{17}$$

and

$$C\colon x^2 + y^2 = 1 \tag{18}$$

(oriented counter-clockwise). Find the flow of $\vec{F}$ along C and the flux of $\vec{F}$ through C.

Solution. We can parameterize the unit circle C by

$$\vec{r}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}}, \qquad t \in [0, 2\pi]. \tag{19}$$

The flow is then given by

$$\begin{aligned}\text{Flow} &= \oint_C \vec{F} \cdot \hat{\mathbf{T}}\,ds \\ &= \oint_C \vec{F} \cdot d\vec{r} \left(= \oint_C M\,dx + N\,dy \right) \\ &= \int_0^{2\pi} \Big((\cos t - \sin t)\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} \Big) \cdot \Big(-\sin t\,\hat{\mathbf{i}} + \cos t\,\hat{\mathbf{j}} \Big)\,dt \\ &= \int_0^{2\pi} \Big(\sin t(\sin t - \cos t) + \cos^2 t \Big)\,dt \\ &= \ldots = 2\pi. \qquad [\textbf{Exercise}: \text{Check this}]\end{aligned} \tag{20}$$

The flux can be computed to be

$$\begin{aligned}\text{Flux} &= \oint_C \vec{F} \cdot \hat{\mathbf{n}}\,ds \\ &= \oint_C M\,dy - N\,dx \\ &= \int_0^{2\pi} (\cos t - \sin t)\,d(\sin t) - \cos t\,d(\cos t) \\ &= \int_0^{2\pi} \Big(\cos t(\cos t - \sin t) + \sin t \cos t \Big)\,dt \\ &= \ldots = \pi. \qquad [\textbf{Exercise}: \text{Check this}]\end{aligned} \tag{21}$$

□

Remark 11. If C is an oriented curve, we usually use $-C$ to denote the same curve with the *opposite orientation.*

If f is a scalar function, we see that

$$\int_C f\,ds = \int_{-C} f\,ds. \tag{22}$$

Here, the orientation does not matter since the quantity ds is not oriented (it just measures "length").

If $\vec{F}$ is a vector field, then

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\,ds = -\int_{-C} \vec{F} \cdot \hat{\mathbf{T}}\,ds. \tag{23}$$

Hence the direction of the curve affects the flow. The reason for the minus sign here is because the orientation of the curve determines the direction of the unit tangent vector $\hat{\mathbf{T}}$. (Recall that $\hat{\mathbf{T}}$ is calculated using a choice of parameterization and so depends on the orientation of C.)

However, for flux, we have

$$\oint_C \vec{F}\cdot\hat{\mathbf{n}}\,ds = \oint_{-C} \vec{F}\cdot\hat{\mathbf{n}}\,ds. \tag{24}$$

Here the direction does not matter since $\hat{\mathbf{n}}$ is *always* taken to be the *outward* unit normal (regardless of orientation).

In summary, we have the following table:

Scalar f	$\int_C f\,ds$ is independent of the orientation of C	ds has no direction
Vector Field $\vec{F}$	$\int_C \vec{F}\cdot\hat{\mathbf{T}}\,ds$ depends on the orientation of C	$\hat{\mathbf{T}}$ depends on the direction
Vector Field $\vec{F}$	$\int_C \vec{F}\cdot\hat{\mathbf{n}}\,ds$ is independent of the orientation of C	$\hat{\mathbf{n}}$ is always outward

(25)

0.0.2 Conservative Vector Fields

Definition 12 (Conservative Vector Field)**.** Let $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) be open. A vector field $\vec{F}$ on Ω is *conservative* if the integral

$$\int_C \vec{F}\cdot\hat{\mathbf{T}}\,ds = \int_C \vec{F}\cdot d\vec{r} \tag{26}$$

along an *oriented* curve C in Ω *depends only* on the *start and end points* of C.

Remark 13 (Path Independence)**.** These are also referred to as *path independent vector fields* since it does not matter which curve C we choose. That is, if C_1 and C_2 are *oriented* curves with the same *start and end points* A and B respectively, then

$$\int_{C_1} \vec{F}\cdot\hat{\mathbf{T}}\,ds = \int_{C_2} \vec{F}\cdot\hat{\mathbf{T}}\,ds. \tag{27}$$

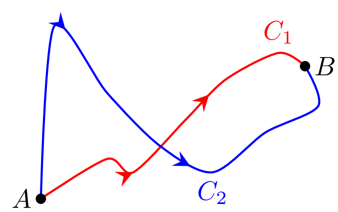


Figure 2: If $\vec{F}$ is conservative, then the integral $\int_C \vec{F}\cdot\hat{\mathbf{T}}\,ds$ will be the same for both curves C_1 and C_2.

In this case, we sometimes write

$$\int_A^B \vec{F}\cdot\hat{\mathbf{T}}\,ds \tag{28}$$

to denote the *common value* of the integral along *any* oriented curve C from A to B.

Example 14. Let $\vec{F} \equiv \hat{\mathbf{i}}$ on $\mathbb{R}^2$ and

$$C : \vec{r}(t) = x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}}, \qquad t \in [a, b]. \tag{29}$$

Then

$$\begin{aligned}\int_C \vec{F}\cdot\hat{\mathbf{T}}\,ds &= \int_C \vec{F}\cdot d\vec{r}\\ &= \int_a^b \hat{\mathbf{i}}\cdot\Big(x'(t)\,\hat{\mathbf{i}} + y'(t)\,\hat{\mathbf{j}}\Big)\,dt\\ &= \int_a^b x'(t)\,dt\\ &= x(b) - x(a).\end{aligned} \tag{30}$$

We notice that $x(a)$ and $x(b)$ are the x-coordinates at $\vec{r}(a)$ and $\vec{r}(b)$ respectively. Thus the integral $\int_C \vec{F}\cdot\hat{\mathbf{T}}\,ds$ depends only on the start point $\vec{r}(a)$ and end point $\vec{r}(b)$ of C. Hence $\vec{F}$ is a conservative vector field.

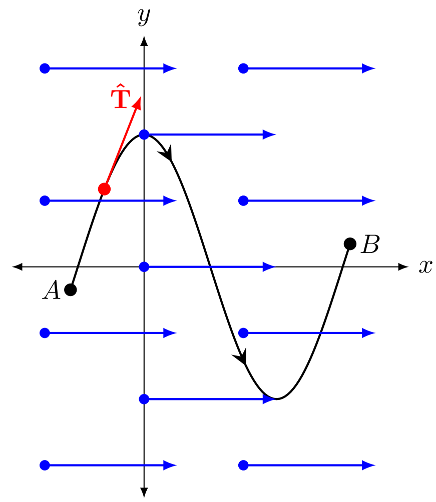


Figure 3: The curve C and vector field $\vec{F}$.

Note that if we let $f(x,y) = x$, then $\vec{F} = \vec{\nabla} f$.

Theorem 15 (Fundamental Theorem of Path Integrals). *Let f be a C^1 function on an open set $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) and let $\vec{F} = \vec{\nabla} f$ be the* gradient vector field *of f. Then, for any piecewise smooth oriented curve C in Ω with start point A and end point B, we have*

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds = f(B) - f(A). \tag{31}$$

Proof. First suppose that C is a smooth curve parameterized by $\vec{r}(t)$ for $t \in [a,b]$. Then

$$\begin{aligned}
\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds &= \int_C \vec{F} \cdot d\vec{r} \\
&= \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)\, dt \\
&= \int_a^b \vec{\nabla} f\big(\vec{r}(t)\big) \cdot \vec{r}\,'(t)\, dt \\
&= \int_a^b \frac{d}{dt} f\big(\vec{r}(t)\big)\, dt && \text{[Chain Rule]} \\
&= f\big(\vec{r}(b)\big) - f\big(\vec{r}(a)\big) && \text{[Fund. Thm. of Calculus]} \\
&= f(B) - f(A).
\end{aligned}$$

If instead C is only piecewise smooth, we can write

$$C = C_1 + C_2 + \ldots + C_m \tag{32}$$

where each C_k is a smooth curve with start point A_k and end point B_k and the + notation indicates that these are joined end-to-end with orientations consistent with C.

Since these are joined end-to-end, we have

$$A_k = B_{k-1} \text{ for } 1 \leq k \leq m \tag{33}$$

and $A_1 = A$ and $B_m = B$.

The previous argument then tells us that

$$\begin{aligned}
\int_C \vec{F}\cdot\hat{\mathbf{T}}\,ds &= \int_{\sum_{k=1}^m C_k} \vec{F}\cdot\hat{\mathbf{T}}\,ds \\
&= \sum_{k=1}^m \int_{C_k} \vec{F}\cdot\hat{\mathbf{T}}\,ds \\
&= \sum_{k=1}^m \Big(f(B_k) - f(A_k) \Big) \\
&= f(B_m) - f(A_m) + f(B_{m-1}) - f(A_{m-1}) + \ldots + f(B_2) - f(A_2) + f(B_1) - f(A_1) \\
&= f(B) - f(A).
\end{aligned}$$

□

Is the converse of the theorem true? Yes (under a condition on the domain Ω).

Theorem 16 (Conservative Potentials on Connected Domains)**.** *Let $\Omega \subseteq \mathbb{R}^n$ ($n = 2$ or 3) be open and* (path) connected *and let $\vec{F}$ be a* continuous *vector field on Ω. Then the following are equivalent:*

i) there exists a C^1 function $f\colon\Omega\to\mathbb{R}$ such that

$$\vec{F} = \vec{\nabla} f, \tag{34}$$

ii) for any closed curve *C on Ω, we have*

$$\oint_C \vec{F}\cdot d\vec{r} = 0, \tag{35}$$

iii) the vector field $\vec{F}$ is conservative.

Remark 17 (Potential Functions)**.** The scalar function f from the above theorem is called the potential function of $\vec{F}$. It is *uniquely determined* up to the addition of a constant since

$$\vec{\nabla}(f + c) = \vec{\nabla} f \tag{36}$$

for any constant c.

Remark 18 (Exact 1-Forms)**.** We can write

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}} = \vec{\nabla} f \iff M\,dx + N\,dy + L\,dz = df \tag{37}$$

(we also have a similar expression in 2 dimensions).

In this case,

$$M\,dx + N\,dy + L\,dz \tag{38}$$

is called an *exact differential (1-)form.*

Remark 19 (Path Connectedness and Connectedness)**.** In general, every path connected set is connected, however the converse is not true. A counter-example is given by the so-called *Topologist's Sine Curve*

$$T = \left\{ \left(x, \sin\frac{1}{x} \right) \middle| x \in (0,1] \right\} \cup \{(0,0)\} \tag{39}$$

For our purposes, since we are working in $\mathbb{R}^n$ ($n = 2$ or 3), every open and connected set is path connected.

(End of Lecture 11 – Oct 13)

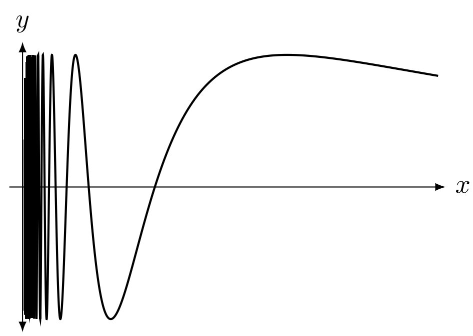


Figure 4: The Topologist's Sine Curve is connected, but not path connected.