MATH2020A Lecture 10 Notes

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Last time, we started looking at vector analysis and defined integration of a scalar function along a curve. We saw that this required a parameterization $\vec{r}\colon [a,b] \to \mathbb{R}^n$ ($n = 2$ or 3) which ultimately turned this integral into a regular 1-dimensional integral:

$$\int_C f(\vec{r})\,ds = \int_a^b f\big(\vec{r}(t)\big) \cdot \|\vec{r'}(t)\|\,dt. \tag{1}$$

Here ds is the arc-length parameter, and $\|\vec{r'}(t)\|$ can be thought of as the length-scaling factor from the Jacobian determinant in a change of variables.

We also took a look at vector fields, which are maps from a set in $\mathbb{R}^n$ to $\mathbb{R}^n$. There is a way to go from a scalar function f to a vector field $\vec{F}$, which is given by the gradient operator:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} \text{ in } \mathbb{R}^2 \text{ or } \vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\,\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\,\hat{\mathbf{k}} \text{ in } \mathbb{R}^3.$$

Example 1 (Vector Field Along a Curve)**.** Let C be a curve in $\mathbb{R}^2$ parameterized by

$$\begin{aligned} \vec{\gamma}\colon [a,b] &\to \mathbb{R}^2 \\ t &\mapsto \big(x(t), y(t)\big) = \vec{\gamma}(t). \end{aligned}$$

We can define the unit tangent vector $\hat{\mathbf{T}}$ along C by

$$\hat{\mathbf{T}} = \frac{\vec{\gamma'}(t)}{\|\vec{\gamma'}(t)\|} \tag{2}$$

wherever $\vec{\gamma'}(t) \neq 0$. Note that $\hat{\mathbf{T}}$ is only defined along C and not outside C. Additionally, if the curve C intersects itself, some points may have more than one tangent vector associated to it.

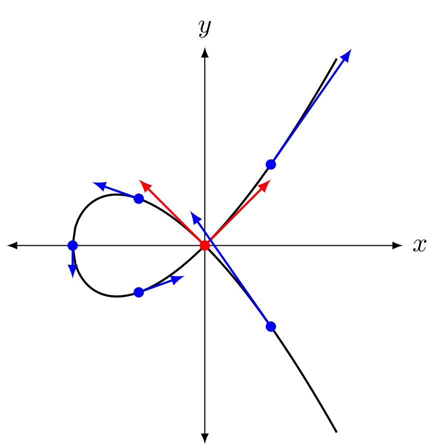

Figure 1: The tangent vector field to the curve parameterized by $\vec{\gamma}(t) = (t^2 - 1, t^3 - 1)$ has two vectors at the origin $(0, 0)$.

A vector field along a curve C does not necessarily have to come from a vector field on a domain D containing C.

Remark 2 (Arc-Length Parameterization)**.** Writing

$$ds = \|\vec{\gamma}'(t)\|\, dt \tag{3}$$

we have

$$\hat{\mathbf{T}} = \frac{\vec{\gamma}'(t)}{\|\vec{\gamma}'(t)\|} = \frac{\frac{d\vec{\gamma}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{\gamma}}{ds} \tag{4}$$

where the *arc-length* s is defined by

$$s(t) = \int_{t_0}^{t} \|\vec{\gamma}'(\tau)\|\, d\tau. \tag{5}$$

A parameterization $\vec{\gamma}(t)$ of a curve C is called an *arc-length parameterization* if

$$s(t) = t - t_0, \tag{6}$$

which implies that

$$\|\vec{\gamma}'(t)\| \equiv 1. \tag{7}$$

Definition 3. A vector field is said to be *continuous/ differentiable/* C^k if its component functions are.

Example 4. The vector field

$$\vec{F}(x,y) = \vec{r}(x,y) = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}} \tag{8}$$

is C^∞ on $\mathbb{R}^{\nvDash}$ whereas the vector field

$$\vec{F}(x,y) = \frac{-y\,\hat{\mathbf{i}} + x\,\hat{\mathbf{j}}}{\sqrt{x^2+y^2}} \tag{9}$$

is *not* continuous on $\mathbb{R}^2$ (but is on $\mathbb{R}^2 \backslash \{(0,0)\}$).

Definition 5 (Path Integral of a Vector Field)**.** Let C be a curve with an "orientation" given by a parameterization $\vec{r}(t)$, where $\vec{r}'(t) \neq 0$ for all t. Let $\vec{F}$ be a vector field defined on C. We define the *path (or line) integral of* $\vec{F}$ *over* C to be

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds \tag{10}$$

where $\hat{\mathbf{T}} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|}$ is the unit tangent vector field along C. (The curve C is oriented in the *direction* of $\vec{r}'(t)$ or $\hat{\mathbf{T}}$ at every point.)

Conceptually, the dot product

$$\vec{F} \cdot \hat{\mathbf{T}} \tag{11}$$

picks out the "amount" of $\vec{F}$ (recall that the dot product returns a scalar, not a vector) that is tangent to the curve and so the integral

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds \tag{12}$$

can be thought of as adding up the "amount" of vector field that is tangent to the curve.

One can check that the quantity

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds \tag{13}$$

is invariant under *orientation-preserving changes of parameter*. If one reverses the orientation, then we have to multiply by a factor of -1.

Remark 6. If $\vec{r}\colon [a,b] \to \mathbb{R}^n$ (where $n = 2$ or 3), then

$$\begin{aligned}\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds &= \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} \|\vec{r}'(t)\|\, dt \\ &= \int_a^b \vec{F}\big(\vec{r}(t)\big) \cdot \vec{r}'(t)\, dt. \end{aligned} \tag{14}$$

As such, we usually denote

$$d\vec{r} = \hat{\mathbf{T}}\, ds \tag{15}$$

and

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds = \int_C \vec{F} \cdot d\vec{r}. \tag{16}$$

Example 7. Let

$$\vec{F}(x, y, z) = z\,\hat{\mathbf{i}} + xy\,\hat{\mathbf{j}} - y^2\,\hat{\mathbf{k}} \tag{17}$$

and C be the curve parameterized by

$$\vec{r}(t) = t^2\,\hat{\mathbf{i}} + t\,\hat{\mathbf{j}} + \sqrt{t}\,\hat{\mathbf{k}}, \qquad t \in [0, 1]. \tag{18}$$

Compute the path integral of $\vec{F}$ along C.

Solution. Computing directly, we can check that

$$d\vec{r} = \left(2t\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \frac{1}{2\sqrt{t}}\,\hat{\mathbf{k}}\right) dt \tag{19}$$

and so

$$\begin{aligned}
\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds &= \int_C \vec{F} \cdot d\vec{r} \\
&= \int_0^1 \left(\sqrt{t}\,\hat{\mathbf{i}} + t^3\,\hat{\mathbf{j}} - t^2\,\hat{\mathbf{k}}\right) \cdot \left(2t\,\hat{\mathbf{i}} + \hat{\mathbf{j}} + \frac{1}{2\sqrt{t}}\,\hat{\mathbf{k}}\right) dt \\
&= \int_0^1 \left(2t\sqrt{t} + t^3 - \frac{t\sqrt{t}}{2}\right) dt \\
&= \left(\frac{3t^{\frac{5}{2}}}{5}n + \frac{t^4}{4}\right)\Bigg|_{t=0}^{t=1} \\
&= \frac{17}{20}.
\end{aligned}$$

□

In components, the path integral of

$$\vec{F}(x, y) = M(x, y)\,\hat{\mathbf{i}} + N(x, y)\,\hat{\mathbf{j}} \tag{20}$$

along

$$C\colon \vec{r}(t) = g(t)\,\hat{\mathbf{i}} + h(t)\,\hat{\mathbf{j}} \tag{21}$$

can be expressed as

$$\begin{aligned}
\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds = \int_C \vec{F} \cdot d\vec{r} &= \int_a^b \left(\vec{F} \cdot \frac{d\vec{r}}{dt}\right) dt \\
&= \int_a^b \left(Mg' + N'h\right) dt.
\end{aligned} \tag{22}$$

More explicitly, we could write

$$\int_a^b \Big[M\big(g(t), h(t)\big)g'(t) + N\big(g(t), h(t)\big)h'(t)\Big]\, dt. \tag{23}$$

Note that

$$\begin{cases} x = g(t), \\ y = h(t), \end{cases} \implies \begin{cases} dx = g'(t)\, dt, \\ dy = h(t)\, dt \end{cases} \tag{24}$$

and so

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_a^b M\, dx + N\, dy. \tag{25}$$

Similarly, in 3 dimensions, we can write

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_a^b M\, dx + N\, dy + L\, dz \tag{26}$$

for

$$\vec{F} = M\,\hat{\mathbf{i}} + N\,\hat{\mathbf{j}} + L\,\hat{\mathbf{k}}. \tag{27}$$

Another way to justify this notation is that if

$$\vec{r} = (x, y, z) \tag{28}$$

is the position vector, then

$$d\vec{r} = (dx, dy, dz) \tag{29}$$

and so

$$\int_C \vec{F} \cdot \hat{\mathbf{T}}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_C (M, N, L) \cdot (dx, dy, dz) = \int_C M\, dx + N\, dy + L\, dz. \tag{30}$$

Example 8. Evaluate the integral

$$I = \int_C -y\, dx + z\, dy + 2x\, dz \tag{31}$$

where C is the curve

$$C\colon \vec{r}(t) = \cos t\,\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + t\,\hat{\mathbf{k}}, \qquad t \in [0, 2\pi]. \tag{32}$$

Solution. We compute that

$$\begin{aligned} I &= \int_C (-\sin t)\, d(\cos t) + t\, d(\sin t) + 2\cos t\, dt \\ &= \int_0^{2\pi} \left(\sin^2 t + t\cos t + 2\cos t\right) dt \\ &= \ldots = \pi. \qquad \text{[Exercise : Check this]} \end{aligned} \tag{33}$$

□

(End of Lecture 10 – Oct 9)