

THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2025-2026 Term 1
Suggested Solutions of WeBWork Coursework 7

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1. Find the maximum area of a triangle formed in the first quadrant by the x -axis, y -axis and a tangent line to the graph of $f(x) = (x + 5)^{-2}$.

Solution:

First, we find the derivative of $f(x)$:

$$f'(x) = \frac{-2}{(x + 5)^3}$$

Let the point of tangency be $(a, f(a))$, where $a \geq 0$. The coordinates are $(a, \frac{1}{(a+5)^2})$ and the slope is $m = f'(a) = \frac{-2}{(a+5)^3}$. The equation of the tangent line in point-slope form is:

$$y - \frac{1}{(a + 5)^2} = \frac{-2}{(a + 5)^3}(x - a)$$

which then gives the x, y -intercepts

$$x_0 = \frac{3a + 5}{2}$$
$$y_0 = \frac{3a + 5}{(a + 5)^3}$$

The area is given by

$$A(a) = \frac{1}{2}x_0y_0 = \frac{(3a + 5)^2}{4(a + 5)^3}$$
$$\implies A'(a) = \frac{3(3a + 5)(5 - a)}{4(a + 5)^4}$$

Set $A'(a) = 0$, we have $a = 5$. For $0 < a < 5$, $A'(a) > 0$ and $A(a)$ is increasing; $a > 5$, $A'(a) < 0$ and $A(a)$ is decreasing. Therefore the maximum value is $A(5) = \frac{1}{10}$.

2. Find the point (x, y) of $x^2 + 14xy + 49y^2 = 100$ that is closest to the origin and lies in the first quadrant.

Solution:

The curve is $(x + 7y)^2 = 100$, or $x + 7y = 10$ in the first quadrant.

Minimize $D = x^2 + y^2$.

$$x = 10 - 7y \implies D(y) = (10 - 7y)^2 + y^2 = 50y^2 - 140y + 100$$

$$D'(y) = 100y - 140 = 0 \implies y = \frac{7}{5}$$

$$x = 10 - 7\left(\frac{7}{5}\right) = \frac{1}{5}$$

For $0 < y < \frac{7}{5}$, $D'(y) < 0$; for $\frac{7}{5} < y$, $D'(y) > 0$. Thus D attains the minimum at $y = \frac{7}{5}$. The final answer is $x = \frac{1}{5}$, $y = \frac{7}{5}$.

3. Use L'Hôpital's Rule (possibly more than once) to evaluate the following limit

$$\lim_{x \rightarrow \infty} \frac{6x^3 + 13x^2}{8x^3 - 8}.$$

Solution: The functions in the numerator and denominator are differentiable, and limit is an indeterminate form of type $\frac{\infty}{\infty}$

$$\lim_{x \rightarrow \infty} \frac{6x^3 + 13x^2}{8x^3 - 8} = \lim_{x \rightarrow \infty} \frac{18x^2 + 26x}{24x^2}$$

Applying L'Hôpital's Rule again, we get:

$$\lim_{x \rightarrow \infty} \frac{18x^2 + 26x}{24x^2} = \lim_{x \rightarrow \infty} \frac{36x + 26}{48x}$$

Applying L'Hôpital's Rule again, we get:

$$\lim_{x \rightarrow \infty} \frac{36x + 26}{48x} = \frac{36}{48} = \frac{3}{4}.$$

4. Compute $\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{2 \sin x}$.

Solution:

The functions in the numerator and denominator are differentiable, and limit is an indeterminate form of type $\frac{0}{0}$. By L'Hôpital's rule, we have

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{2 \sin x} = \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{2 \cos x} = 1.$$

5. Evaluate $\lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{x^2}{2}}{14x^4}$.

Solution:

Using Taylor expansion, $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - O(x^6)$:

$$\lim_{x \rightarrow 0} \frac{(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots) - 1 + \frac{x^2}{2}}{14x^4} = \lim_{x \rightarrow 0} \frac{\frac{x^4}{24} + \dots}{14x^4} = \frac{1}{336}$$

6. Evaluate $\lim_{x \rightarrow 0} \frac{\ln(1-x) + x + \frac{x^2}{2}}{13x^3}$.

Solution:

Using Taylor expansion, $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - O(x^4)$:

$$\lim_{x \rightarrow 0} \frac{(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots) + x + \frac{x^2}{2}}{13x^3} = \lim_{x \rightarrow 0} \frac{-\frac{x^3}{3} + \dots}{13x^3} = -\frac{1}{39}$$

7. Find the first three nonzero terms of the Taylor series for the function $f(x) = \sqrt{10x - x^2}$ about the point $a = 5$.

Solution:

$$\begin{aligned} f(x) &= \sqrt{10x - x^2} \\ &= \sqrt{25 - (x-5)^2} \\ &= 5\sqrt{1 - \frac{(x-5)^2}{25}} \\ &= 5 \left[1 + \frac{1}{2} \left(-\frac{(x-5)^2}{25} \right) + \frac{\frac{1}{2}(-\frac{1}{2})}{2!} \left(-\frac{(x-5)^2}{25} \right)^2 + \dots \right] \\ &= 5 - \frac{1}{10}(x-5)^2 - \frac{1}{1000}(x-5)^4 + \dots \end{aligned}$$

8. Write the Taylor series for $f(x) = \ln(\sec(x))$ at $x = 0$ as $\sum_{n=0}^{\infty} c_n x^n$. Find the first five coefficients.

Solution:

- $f(x) = \ln(\sec x) \implies f(0) = \ln 1 = 0 \implies c_0 = 0$
- $f^{(1)}(x) = \tan x \implies f^{(1)}(0) = \tan 0 = 0 \implies c_1 = 0$
- $f^{(2)}(x) = \sec^2 x \implies f^{(2)}(0) = \sec^2 0 = 1 \implies c_2 = 1/2! = 1/2$
- $f^{(3)}(x) = 2 \sec^2 x \tan x \implies f^{(3)}(0) = 2 \sec^2 0 \tan 0 = 0 \implies c_3 = 0$
- $f^{(4)}(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x \implies f^{(4)}(0) = 2 \implies c_4 = 2/4! = 1/12$

9. Write the Taylor series for $f(x) = \sin x$ at $x = \frac{\pi}{2}$ as $\sum_{n=0}^{\infty} c_n (x - \frac{\pi}{2})^n$. Find the first five coefficients.

Solution:

Let $u = x - \pi/2$. Then $\sin(x) = \sin(u + \pi/2) = \cos(u)$.

$$\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} - \dots = 1 - \frac{1}{2!} \left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!} \left(x - \frac{\pi}{2}\right)^4 - \dots$$

Hence we have

$$c_0 = 1, \quad c_1 = 0, \quad c_2 = -1/2, \quad c_3 = 0, \quad c_4 = 1/24.$$