THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2025-2026 Term 1
Suggested Solutions of WeBWork Coursework 6

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Question 1

Let $f(x) = 6\sqrt{x} - 4x$ for $x > 0$. Find the intervals on which f is increasing/decreasing.

Solution:

1. Find the derivative:
$$f'(x) = \frac{d}{dx}(6x^{1/2}) - \frac{d}{dx}(4x) = 6 \cdot \frac{1}{2}x^{-1/2} - 4 = \frac{3}{\sqrt{x}} - 4$$
2. Set derivative equal to zero:
$$\frac{3}{\sqrt{x}} - 4 = 0 \Rightarrow \frac{3}{\sqrt{x}} = 4 \Rightarrow \sqrt{x} = \frac{3}{4} \Rightarrow x = \left(\frac{3}{4}\right)^2 = \frac{9}{16} = 0.5625$$
3. Test intervals:
For $0 < x \leq 0.5625$: $\sqrt{x} \leq 0.75 \Rightarrow \frac{3}{\sqrt{x}} \geq 4 \Rightarrow f'(x) \geq 0$. Then f is increasing;
For $x \geq 0.5625$: $\sqrt{x} \geq 0.75 \Rightarrow \frac{3}{\sqrt{x}} \leq 4 \Rightarrow f'(x) \leq 0$. Then f is decreasing.

Final Answer:

$$\text{Increasing: } (0, 0.5625], \quad \text{Decreasing: } [0.5625, \infty)$$

Question 2

Find the critical point and the interval on which the given function is increasing or decreasing. Let $f(x) = x - 2\ln(6x), x > 0$.

Solution:

1. First derivative:
$$f'(x) = 1 - 2 \cdot \frac{1}{x} = 1 - \frac{2}{x}$$
2. Critical point:
$$1 - \frac{2}{x} = 0 \Rightarrow \frac{2}{x} = 1 \Rightarrow x = 2$$
3. Second derivative test:
$$f''(x) = \frac{2}{x^2} \Rightarrow f''(2) = \frac{2}{4} = 0.5 > 0 \Rightarrow \text{Local minimum}$$
4. Interval analysis:
   - $(0, 2)$: f' is negative $\Rightarrow$ f is decreasing,
   - $(2, \infty)$: f' is positive $\Rightarrow$ f is increasing.

Final Answer: Critical point: 2, Local Min, $(0, 2)$, Decreasing, Negative, $(2, \infty)$, Increasing, Positive.

Question 3

Determine concavity and inflection point for $f(x) = x(x - 8\sqrt{x})$.

Solution:

1. Simplify function:
$$f(x) = x^2 - 8x^{3/2}$$
2. First derivative:
$$f'(x) = 2x - 8 \cdot \frac{3}{2}x^{1/2} = 2x - 12\sqrt{x}$$
3. Second derivative:
$$f''(x) = 2 - 12 \cdot \frac{1}{2\sqrt{x}} = 2 - \frac{6}{\sqrt{x}}$$
4. x-coordinate of the inflection point:
$$2 - \frac{6}{\sqrt{x}} = 0 \Rightarrow \frac{6}{\sqrt{x}} = 2 \Rightarrow \sqrt{x} = 3 \Rightarrow x = 9$$
5. Concavity test:
   - $(0, 9)$: f'' is negative $\Rightarrow$concave down
   - $(9, \infty)$: f'' is positive $\Rightarrow$ concave up

Final Answer: Inflection at $x = 9$, $(0, 9)$, Concave Down, $(9, \infty)$, Concave Up.

Question 4

For $f(x) = \dfrac{7e^x}{7e^x + 4}$, find critical values and concavity.

Solution:

1. First derivative:

$$f'(x) = \frac{(7e^x)(7e^x + 4) - (7e^x)(7e^x)}{(7e^x + 4)^2} = \frac{28e^x}{(7e^x + 4)^2}$$

   Since $f'(x) > 0$ for all x, no critical points.

2. Second derivative:

$$f''(x) = \frac{28e^x(4 - 7e^x)}{(7e^x + 4)^3}$$

3. Inflection point:

$$4 - 7e^x = 0 \Rightarrow e^x = \frac{4}{7} \Rightarrow x = \ln\left(\frac{4}{7}\right) \approx -0.5596$$

4. Concavity:

   - $x < \ln\left(\frac{4}{7}\right)$: $4 - 7e^x > 0 \Rightarrow$ concave up
   - $x > \ln\left(\frac{4}{7}\right)$: $4 - 7e^x < 0 \Rightarrow$ concave down

Final Answer: Critical values: None, Concave up: $(-\infty, \ln\left(\frac{4}{7}\right))$, Concave down: $(\ln\left(\frac{4}{7}\right), \infty)$, Inflection at $x = \ln\left(\frac{4}{7}\right) \approx -0.5596$.

Question 5

Find extreme values of $f(x) = x^8 + \dfrac{8}{x}$ on the interval $[0.5, 5]$.

Solution:

1. Derivative:
$$f'(x) = 8x^7 - \frac{8}{x^2}$$
2. Critical point:
$$8x^7 = \frac{8}{x^2} \Rightarrow x^9 = 1 \Rightarrow x = 1$$
3. Evaluate function:
$$\begin{aligned} f(0.5) &= (0.5)^8 + \frac{8}{0.5} = 0.00390625 + 16 = 16.00390625 \\ f(1) &= 1 + 8 = 9 \\ f(5) &= 5^8 + \frac{8}{5} = 390625 + 1.6 = 390626.6 \end{aligned}$$

Final Answer: Absolute minimum: 9, Absolute maximum: 390626.6.

Question 6

Analyze $f(x) = \dfrac{(x+5)^2}{(x-5)^2}$.

Solution:

1. Vertical asymptote:
$$x - 5 = 0 \Rightarrow x = 5$$
2. Horizontal asymptote:
$$\lim_{x\to\pm\infty} \frac{(x+5)^2}{(x-5)^2} = \lim_{x\to\pm\infty} \frac{x^2(1+5/x)^2}{x^2(1-5/x)^2} = 1 \Rightarrow y = 1$$
3. Points where the graph crosses a horizontal asymptote:
$$\frac{(x+5)^2}{(x-5)^2} = 1 \Rightarrow (x+5)^2 = (x-5)^2 \Rightarrow 20x = 0 \Rightarrow x = 0 \Rightarrow (0, 1)$$
4. Critical points:
$$f'(x) = \frac{-20(x+5)}{(x-5)^3} = 0 \Rightarrow x = -5 \Rightarrow (-5, 0)$$
5. Inflection points:
$$f''(x) = \frac{40(x+10)}{(x-5)^4} = 0 \Rightarrow x = -10 \Rightarrow \left(-10, \frac{1}{9}\right)$$

Final Answer: (a) 5, (b) 1, (c) $(0, 1)$, (d) $(-5, 0)$, (e) $(-10, 1/9)$.

Question 7

Consider the function $f(x) = x^2 - 4x + 6$ on the interval $[0, 4]$. Verify that this function satisfies the three hypotheses of Rolle's Theorem on the interval.

Solution: Rolle's Theorem has three hypotheses:

1. $f(x)$ is continuous on the closed interval $[a, b]$
2. $f(x)$ is differentiable on the open interval (a, b)
3. $f(a) = f(b)$

Step 1: Verify continuity on $[0, 4]$ The function $f(x) = x^2 - 4x + 6$ is a polynomial function. All polynomial functions are continuous everywhere on $\mathbb{R}$. Therefore, $f(x)$ is continuous on $[0, 4]$.

Step 2: Verify differentiability on $(0, 4)$ The function $f(x) = x^2 - 4x + 6$ is a polynomial function. All polynomial functions are differentiable everywhere on $\mathbb{R}$. Therefore, $f(x)$ is differentiable on $(0, 4)$.

Step 3: Verify $f(0) = f(4)$
Calculate $f(0)$:

$$f(0) = (0)^2 - 4(0) + 6 = 6$$

Calculate $f(4)$:

$$f(4) = (4)^2 - 4(4) + 6 = 16 - 16 + 6 = 6$$

Since $f(0) = 6$ and $f(4) = 6$, we have $f(0) = f(4)$.

Step 4: Find c **such that** $f'(c) = 0$ First, find the derivative:

$$f'(x) = \frac{d}{dx}(x^2 - 4x + 6) = 2x - 4$$

Set the derivative equal to zero:

$$f'(c) = 2c - 4 = 0$$

$$2c = 4$$

$$c = 2$$

Verify that $c = 2$ is in the interval $(0, 4)$: Since $0 < 2 < 4$, $c = 2$ is indeed in the open interval $(0, 4)$.

Final Answers:

- $f(x)$ is **continuous** on $[0, 4]$
- $f(x)$ is **differentiable** on $(0, 4)$
- $f(0) = f(4) = \mathbf{6}$
- $c = \mathbf{2}$

Question 8

Find absolute extrema of $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ on the interval$[-5, 5]$.

Solution:

1. Derivative:
$$f'(x) = \frac{(2x)(x^2 + 1) - (x^2 - 1)(2x)}{(x^2 + 1)^2} = \frac{4x}{(x^2 + 1)^2}$$

2. Critical point:
$$f'(x) = 0 \Rightarrow 4x = 0 \Rightarrow x = 0$$

3. Evaluate:
$$f(0) = \frac{-1}{1} = -1$$
$$f(-5) = f(5) = \frac{25 - 1}{25 + 1} = \frac{24}{26} = \frac{12}{13}$$

Final Answer: Absolute maximum: $\frac{12}{13}$, Absolute minimum: -1.

Question 9

True/False quiz.

Solution:

1. **T** - Sum rule of differentiation: $(f(x) + g(x))' = f'(x) + g'(x)$
2. **T** - Differentiable functions are always continuous (differentiability implies continuity)
3. **T** - A continuous function on a closed interval always attains a maximum and a minimum value (Extreme Value Theorem)

4. **T** - If $f'(c) = 0$ and $f''(c) > 0$, then $f(x)$ has a local minimum at c (Second Derivative Test)

5. **F** - If a function has a local maximum at c, then $f'(c)$ exists and is equal to 0 (counterexample: $f(x) = -|x|$ at $x = 0$)

6. **F** - Continuous functions are always differentiable (counterexample: $f(x) = |x|$ at $x = 0$)

7. **F** - If $f'(c) = 0$, then c is either a local maximum or a local minimum (counterexample: $f(x) = x^3$ at $x = 0$)

Final Answer: T, T, T, T, F, F, F