THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2025-2026 Term 1
Suggested Solutions of WeBWork Coursework 4

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1.

Let

$$f(x) = \begin{cases} -5x, & x < 3, \\ 1, & x = 3, \\ 5x, & x > 3. \end{cases}$$

Find the indicated one-sided limits of f, and determine the continuity of f at the indicated point. You should also sketch a graph of $y = f(x)$, including hollow and solid circles in the appropriate places.

NOTE: Type DNE if a limit does not exist.

$$\lim_{x\to 3^-} f(x) = \boxed{-15}$$
$$\lim_{x\to 3^+} f(x) = \boxed{15}$$
$$\lim_{x\to 3} f(x) = \boxed{\text{DNE}}$$
$$f(3) = \boxed{1}$$

Is f continuous at $x = 3$? $\boxed{\text{NO}}$

Solution

We will compute the one-sided limits as $x \to 3$ from the left and right.

The Left-hand limit: As $x \to 3^-$, we are approaching 3 from values less than 3, so we use the definition $f(x) = -5x$. Then:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (-5x) = -5 \cdot 3 = -15.$$

The Right-hand limit: As $x \to 3^+$, we are approaching 3 from values greater than 3, so we use $f(x) = 5x$. Then:

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (5x) = 5 \cdot 3 = 15.$$

Two-sided limit: Since the left-hand limit (-15) and the right-hand limit (15) are not equal:

$$\lim_{x \to 3} f(x) \text{ does not exist (DNE).}$$

By the definition of $f(x)$, we have: $f(3) = 1$.

A function is continuous at a point $x = a$ if and only if:

1. $f(a)$ is defined,
2. $\lim_{x \to a} f(x)$ exists,
3. $\lim_{x \to a} f(x) = f(a)$.

Here, we have $f(3) = 1$ is defined, but $\lim_{x \to 3} f(x)$ does not exist. So, f is not continuous at $x = 3$.

The graph of $f(x)$, we plot as following:

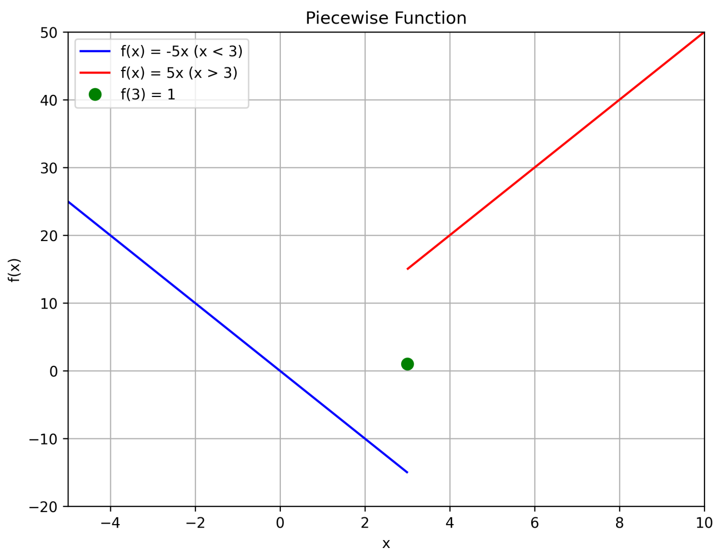

Figure 1: Graph of the piecewise function $f(x)$

2.

Let $f(x) = |x - 5|$. Evaluate the following limits.

$$\lim_{x \to 5^-} \frac{f(x) - f(5)}{x - 5} = \boxed{-1}$$

$$\lim_{x \to 5^+} \frac{f(x) - f(5)}{x - 5} = \boxed{1}$$

Thus the function $f(x)$ is not differentiable at 5.

Solution

From the definition of $f(x)$, we have:

$$f(x) = |x-5| = \begin{cases} 5-x, & x < 5, \\ x-5, & x \geq 5. \end{cases}$$

Also, $f(5) = |5-5| = 0$.

We are to evaluate the left-hand and right-hand derivatives at $x = 5$:

$$\lim_{x \to 5^-} \frac{f(x)-f(5)}{x-5} = \lim_{x \to 5^-} \frac{(5-x)-0}{x-5} = \lim_{x \to 5^-} \frac{5-x}{x-5} = -1.$$

$$\lim_{x \to 5^+} \frac{f(x)-f(5)}{x-5} = \lim_{x \to 5^+} \frac{(x-5)-0}{x-5} = \lim_{x \to 5^+} \frac{x-5}{x-5} = 1.$$

3.

Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Find $f'(x)$ and $f'(0)$.

(a) Find the derivative of $f(x)$ for $x \neq 0$.
$f'(x) = \boxed{2x\sin(1/x) - \cos(1/x)}$

(b) If the derivative does not exist, enter DNE.
$f'(0) = \boxed{0}$

Solution

(a) Let $u = x^2$, so $u' = 2x$, $v = \sin\left(\frac{1}{x}\right)$, so $v' = \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$

Then by the product rule:

$$f'(x) = u'v + uv' = (2x)\sin\left(\frac{1}{x}\right) + x^2\left(-\frac{1}{x^2}\cos\left(\frac{1}{x}\right)\right)$$

Simplify:

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

(b) We use the definition of the derivative:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right).$$

Note that:

$$\left|\sin\left(\frac{1}{h}\right)\right| \le 1 \quad \text{for all } h \neq 0$$

so:

$$\left|h\sin\left(\frac{1}{h}\right)\right| \le |h|$$

As $h \to 0$, $|h| \to 0$, so we have

$$f'(0) = \lim_{h\to 0} h\sin\left(\frac{1}{h}\right) = 0$$

4.

Let

$$f(x) = \begin{cases} -7x^2 + 3x, & x < 0 \\ 7x^2 - 2, & x \ge 0 \end{cases}$$

According to the definition of the derivative, to compute $f'(0)$, we need to compute the left-hand limit

$$\lim_{h\to 0^-} \boxed{\frac{f(0+h) - f(0)}{h}} \text{, which is } \boxed{-\infty},$$

and the right-hand limit

$$\lim_{h\to 0^+} \boxed{\frac{f(0+h) - f(0)}{h}} \text{, which is } \boxed{0}.$$

We conclude that $f'(0)$ is $\boxed{\text{DNE}}$.

Solution

When $h \to 0^-$, $h < 0$, so we use the definition for $x < 0$, then:

$$\frac{f(0+h) - f(0)}{h} = \frac{(-7h^2 + 3h) - (-2)}{h} = \frac{-7h^2 + 3h + 2}{h}.$$

So,

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \left(-7h + 3 + \frac{2}{h}\right) = -\infty$$

The term $\frac{2}{h}$ goes to $-\infty$ as $h \to 0^-$, so the left-hand limit does not exist.

When $h \to 0^+$, $h > 0$, so we use the definition for $x \ge 0$, then:

$$\frac{f(0+h) - f(0)}{h} = \frac{(7h^2 - 2) - (-2)}{h} = \frac{7h^2}{h} = 7h.$$

So,

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} 7h = 0.$$

Since the left-hand limit does not exist and the right-hand limit is 0, we conclude that $f'(0)$ does not exist.

5.

Evaluate the following limits. If needed, enter 'INF' for ∞ and '-INF' for $-\infty$.

$$(a) \lim_{x\to\infty} \frac{\sqrt{11+9x^2}}{2+11x} = \boxed{\frac{3}{11}}$$

$$(b) \lim_{x\to-\infty} \frac{\sqrt{11+9x^2}}{2+11x} = \boxed{-\frac{3}{11}}$$

Solution

(a) For $x \to \infty$, we divide both the numerator and denominator by x (since $x > 0$):

$$\frac{\sqrt{11+9x^2}}{2+11x} = \frac{\frac{\sqrt{11+9x^2}}{x}}{\frac{2+11x}{x}} = \frac{\sqrt{\frac{11}{x^2}+9}}{\frac{2}{x}+11}$$

Now, take the limit as $x \to \infty$:

$$\lim_{x\to\infty} \frac{\sqrt{\frac{11}{x^2}+9}}{\frac{2}{x}+11} = \frac{\sqrt{0+9}}{0+11} = \frac{\sqrt{9}}{11} = \frac{3}{11}$$

Thus, the limit is $\frac{3}{11}$.

(b) For $x \to -\infty$, we note that $x < 0$. To handle the square root properly, we divide both the numerator and denominator by $-x$ (since $-x > 0$):

$$\frac{\sqrt{11+9x^2}}{2+11x} = \frac{\frac{\sqrt{11+9x^2}}{-x}}{\frac{2+11x}{-x}} = \frac{\sqrt{\frac{11}{x^2}+9}}{-\frac{2}{x}-11}$$

Now, take the limit as $x \to -\infty$:

$$\lim_{x\to-\infty} \frac{\sqrt{\frac{11}{x^2}+9}}{-\frac{2}{x}-11} = \frac{\sqrt{0+9}}{0-11} = \frac{\sqrt{9}}{-11} = \frac{3}{-11} = -\frac{3}{11}$$

Thus, the limit is $-\frac{3}{11}$.

6.

Find a and b so that the function

$$f(x) = \begin{cases} 5x^3 - 2x^2 + 3, & x < -2, \\ ax + b, & x \geq -2 \end{cases}$$

is both continuous and differentiable.

$$a = \boxed{68}, \quad b = \boxed{91}.$$

Solution

For f is continuous at $x = -2$, we need:

$$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^+} f(x) = f(-2).$$

Calculating the left-hand limit:

$$\lim_{x \to -2^-} f(x) = 5(-2)^3 - 2(-2)^2 + 3 = -40 - 8 + 3 = -45.$$

Calculating the right-hand limit:

$$\lim_{x \to -2^+} f(x) = a(-2) + b = -2a + b.$$

Setting the limits equal for continuity:

$$-45 = -2a + b. \quad (1)$$

For f to be differentiable at $x = -2$, we need:

$$\lim_{x \to -2^-} f'(x) = \lim_{x \to -2^+} f'(x).$$

Calculating the left-hand derivative at -2:

$$\lim_{x \to -2^-} f'(x) = 15(-2)^2 - 4(-2) = 60 + 8 = 68.$$

Calculating the right-hand derivative at -2:

$$\lim_{x \to -2^+} f'(x) = a.$$

Setting the derivatives equal for differentiability:

$$68 = a. \quad (2)$$

Substituting $a = 68$ into equation (1):

$$-45 = -2(68) + b \implies -45 = -136 + b \implies b = -45 + 136 = 91.$$

Thus, the values of a and b that make f both continuous and differentiable at $x = -2$ are:

$$a = 68, \quad b = 91.$$

7.

Suppose $f'(x)$ exists for all x in (a, b). Mark all true items with a check. There may be more than one correct answer.

Options:

- A. $f(x)$ is continuous on (a, b).
- B. $f(x)$ is continuous at $x = a$.
- C. $f(x)$ is defined for all x in (a, b).
- D. $f'(x)$ is differentiable on (a, b).

Solition

Answer: $\boxed{AC}$.

Since $f'(x)$ exists for all x in (a, b), this means that $f(x)$ is differentiable on the open interval (a, b).

- **Option A:** If a function is differentiable on an interval, it must be continuous on that interval. Therefore, $f(x)$ is continuous on (a, b). Hence, **A is true**.
- **Option B:** The differentiability of $f(x)$ on (a, b) does not imply continuity at the endpoint $x = a$, because a may not be in the open interval. For example, $f(x)$ might not be defined or continuous at $x = a$. Thus, **B is false**.
- **Option C:** For $f'(x)$ to exist at every point in (a, b), $f(x)$ must be defined at every point in (a, b). Therefore, **C is true**.
- **Option D:** The existence of $f'(x)$ on (a, b) does not guarantee that $f'(x)$ itself is differentiable (i.e., that $f''(x)$ exists). So, **D is false**.

8.

If $f'(a)$ exists, then what can be said about the limit

$$\lim_{x \to a} f(x)?$$

A. must exist, but there is not enough information to determine its value.

B. is equal to $f(a)$.

C. is equal to $f'(a)$.

D. might not exist.

E. does not exist.

Solution

We are given that the derivative $f'(a)$ exists. Recall the definition of the derivative:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

For this limit to exist, the function f must be defined in a neighborhood around a, and the difference quotient must approach a finite number.

However, the existence of $f'(a)$ implies more than just the existence of the derivative, it implies that f is continuous at $x = a$ and the value of limitation is equal to $f(a)$.

Thus, the correct answer is:

$$\boxed{\text{B. } \lim_{x \to a} f(x) \text{ is equal to } f(a).}$$

9.

Given the function

$$h(k) = \sqrt{7-k} + \sqrt{k+3},$$

find the intervals on which $h(k)$ is continuous. Use interval notation with 'INF' for ∞, '-INF' for $-\infty$, and $\cup$ for union.

$\boxed{[-3, 7]}$

Solution

The function $h(k) = \sqrt{7-k} + \sqrt{k+3}$ consists of two square root terms. To determine the intervals on which $h(k)$ is continuous, we need to find the domain of each square root term.

For the term $\sqrt{7-k}$, the expression inside the square root must be non-negative:

$$7 - k \geq 0 \implies k \leq 7.$$

Therefore, k can take any value less than or equal to 7.

For the term $\sqrt{k+3}$, the expression inside the square root must also be non-negative:

$$k + 3 \geq 0 \implies k \geq -3.$$

Therefore, k can take any value greater than or equal to -3.

Combining these two conditions, we find that $h(k)$ is defined and continuous on the interval:

$$[-3, 7].$$

10.

Match each of the following statements to the function (A–F) that best satisfies it.

1. $\boxed{D}$ $\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4^-} f(x)$ both exist and are finite, but they are not equal.
2. $\boxed{E}$ The graph of $y = f(x)$ has a vertical tangent line at $(4, f(4))$.
3. $\boxed{C}$ $\lim_{x\to 4^-} f(x) = -\infty$.
4. $\boxed{B}$ $\lim_{x\to 4^+} f(x)$ exists but $\lim_{x\to 4^-} f(x)$ does not.
5. $\boxed{A}$ $\lim_{x\to 4} f(x) = \infty$.
6. $\boxed{F}$ $\lim_{x\to 4} f(x)$ exists but f is not continuous at 4.

A. $f(x) = \dfrac{1}{(x-4)^2}$

B. $f(x) = \begin{cases} \cos\left(\frac{1}{x-4}\right), & x < 4 \\ 0, & x = 4 \\ 3x + 24, & x > 4 \end{cases}$

C. $f(x) = \dfrac{1}{x-4}$

D. $f(x) = \begin{cases} 3x, & x < 4 \\ 0, & x = 4 \\ 3x - 24, & x > 4 \end{cases}$

E. $f(x) = \sqrt[3]{x-4}$

F. $f(x) = \begin{cases} 3x, & x < 4 \\ 0, & x = 4 \\ 24 - 3x, & x > 4 \end{cases}$

Solution

- **D**: The left limit is $\lim_{x\to 4^-} = \lim_{x\to 4^-} 3x = 12$, and the right limit is$\lim_{x\to 4^+} = \lim_{x\to 4^+} 3x - 24 = -12$: both finite, unequal $\to$ Statement 1.
- **E**: $f(x)$ is defined on $(-\infty, 4]$, giving vertical tangent $\to$ Statement 2.
- **C**: As $x \to 4^-$, $x - 4$ approaches 0^-, so $f(x) \to -\infty \to$ Statement 3.
- **B**: Left-hand limit oscillates and does not exist; right-hand limit is $\lim_{x\to 4+}(3x + 24) = 36 \to$ Statement 4.
- **A**: Both one-sided limits go to $+\infty \to$ overall limit is $\infty \to$ Statement 5.
- **F**: The limit $\lim_{x=4} f(x) = 12$, but $f(4) = 0 \to$ limit exists but function is not continuous $\to$ Statement 6.

11.

Why is the following function discontinuous at $x = 0$?

$$f(x) = \begin{cases} e^x, & if\ x < 0, \\ x^2, & if\ x \geq 0. \end{cases}$$

Choose the correct reason:

1. $f(0)$ does not exist.
2. $\lim\limits_{x\to 0} f(x)$ does not exist (or is infinite).

3. Both (a) and (b).

4. $f(0)$ and $\lim_{x \to 0} f(x)$ exist, but they are not equal.

Solution

Answer: $\boxed{b}$.

Since $x = 0$ satisfies $x \geq 0$, we use the second case:

$$f(0) = 0^2 = 0.$$

So $f(0)$ **exists** and equals 0.

Thus, option (a) is false.

We compute the left-hand and right-hand limits. As $x \to 0^-$: $f(x) = e^x$, so

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} e^x = e^0 = 1.$$

As $x \to 0^+$: $f(x) = x^2$, so

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0.$$

Since the left-hand limit (1) and right-hand limit (0) are not equal,

$$\lim_{x \to 0} f(x) \quad \text{does not exist.}$$

So option (b) is true.