THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2025-2026 Term 1
Suggested Solutions of WeBWork Coursework 1

If you find any errors or typos, please email us at
math1010@math.cuhk.edu.hk

Question 1

In each part, find a formula for the general term of the sequence, starting with $n = 1$.

Enter the following information for $a_n =$.

(a)

$$\frac{1}{36}, \frac{1}{216}, \frac{1}{1296}, \frac{1}{7776}, \cdots$$

$a_n =$ ________

(b)

$$\frac{1}{36}, -\frac{1}{216}, \frac{1}{1296}, -\frac{1}{7776}, \cdots$$

$a_n =$ ________

(c)

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots$$

$a_n =$ ________

(d)

$$0, \frac{1}{\sqrt{\pi}}, \frac{4}{\sqrt[3]{\pi}}, \frac{9}{\sqrt[4]{\pi}}, \cdots$$

$a_n =$ ________

(a) $a_n = \dfrac{1}{6^{n+1}}$ (b) $a_n = \dfrac{(-1)^{n+1}}{6^{n+1}}$ (c)$a_n = \dfrac{2^n - 1}{2^n}$ (d) $a_n = \dfrac{(n-1)^2}{\sqrt[n]{\pi}}$

Question 2

Determine whether the sequence $a_n = \dfrac{n^{14} + \sin(16n + 11)}{n^{16} + 11}$ converges or diverges. If it converges, find the limit.

Converges (y/n): y
Limit: 0

Explanation: Since $\dfrac{n^{14}}{n^{16}} = \dfrac{1}{n^2} \to 0$ as $n \to \infty$ and $|\sin(16n + 11)| \leq 1$, the numerator is dominated by n^{14} while the denominator is dominated by n^{16}, so the limit is 0.

Question 3

Use algebra to simplify the expression before evaluating the limit. In particular, factor the highest power of n from the numerator and denominator, then cancel as many factors of n as possible. If the sequence does not converge, enter *DNE* in the final answer box.

$$\lim_{n\to\infty} \frac{2n}{(5n^3+7)^{1/3}}$$

$$\lim_{n\to\infty} \frac{2n}{(5n^3+7)^{1/3}} = \lim_{n\to\infty} \frac{2n}{n(5+\frac{7}{n^3})^{1/3}} = \lim_{n\to\infty} \frac{2}{(5+\frac{7}{n^3})^{1/3}} = \frac{2}{\sqrt[3]{5}}$$

Question 4

Part 1: Evaluating a series

Consider the sequence $\{a_n\} = \left\{\frac{2}{n^2+2n}\right\}$.

a. The limit of this sequence is $\lim_{n\to\infty} a_n =$ ______.

b. The sum of all terms in this sequence is defined as the the limit of the partial sums, which means

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}\left(\_\_\_\_\_\_\_\_ \right) = \_\_\_\_.$$

a. $\lim_{n\to\infty} a_n = 0$

b. $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{2}{k(k+2)} = \lim_{n\to\infty} \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} = \frac{3}{2}$

(Using telescoping series: $\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$)

Part 2: Evaluating another series

Consider the sequence $\{b_n\} = \left\{\ln\left(\frac{n+1}{n}\right)\right\}$.

a. The limit of this sequence is $\lim_{n\to\infty} b_n =$ ______.

b. The sum of all terms in this sequence is defined as the the limit of the partial sums, which means

$$\sum_{n=1}^{\infty} b_n = \lim_{n\to\infty}\left(\_\_\_\_\_\_\_ \right) = \_\_\_\_.$$

a. $\lim_{n\to\infty} b_n = 0$

b. $\sum_{n=1}^{\infty} b_n = \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(\frac{k+1}{k}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} (\ln(k+1) - \ln k) = \lim_{n\to\infty} \ln(n+1) = \infty$

Part 3: Developing conceptual understanding

Suppose $\{c_n\}$ is a sequence.

a. If $\lim_{n\to\infty} c_n = 0$, then the series $\sum_{n=1}^{\infty} c_n$ may or may not converge.

b. If $\lim_{n\to\infty} c_n \neq 0$, then the series $\sum_{n=1}^{\infty} c_n$ cannot converge.

c. If the series $\sum_{n=1}^{\infty} c_n$ converges, then $\lim_{n\to\infty} c_n$ must be equal to 0.

Question 5

Consider the recursively defined sequence:

$$a_1 = 7$$

$$a_{n+1} = \frac{n+1}{n^2} a_n, \quad \text{for } n \geq 1$$

The sequence is Eventually monotone decreasing
The sequence is bounded below by 0
The sequence is bounded above by 14
The limit of the sequence is: 0

Explanation: The sequence is eventually monotone decreasing since $a_2 = 14 > 7 = a_1$ while

$$a_{n+1} = \frac{n+1}{n^2} a_n \leq a_n \quad \text{for } n \geq 2.$$

Hence we see that $a_n \leq a_2 = 14$ for $n \geq 1$. So the sequence is bounded above by 14.

Since sequence is eventually monotone decreasing and bounded below, the sequence is convergent by monotone convergence theorem. As both the sequence a_n and expression $b_n := \frac{n+1}{n^2}$ have limit, it gives the result that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{n+1}{n^2} a_n = 0 \cdot \lim_{n\to\infty} a_n = 0$.

Question 6

Consider the recursively defined sequence:

$$a_1 = 1, \quad a_2 = 1$$

$$a_{n+2} = \frac{a_{n+1} + a_n}{2}, \qquad \text{for } n \geq 1$$

The limit of the sequence is: 1

Optional: For different a_1 and a_2, the sequence converges to $\frac{a_1 + 2a_2}{3}$.

Question 7

Consider the sequence

$$a_n = \frac{n\cos(n\pi)}{2n-1}.$$

First five terms: $-1, \frac{2}{3}, -\frac{3}{5}, \frac{4}{7}, -\frac{5}{9}$
$\lim_{n\to\infty} a_n = \text{DNE}$

Explanation: The sequence oscillates between positive and negative values due to $\cos(n\pi) = (-1)^n$, so it does not converge.

Question 8

The sequence $\{a_n\}$ is defined by $a_1 = 2$, and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right),$$

for $n \geqslant 1$. Assuming that $\{a_n\}$ converges, find its limit.
Assuming convergence, let $L = \lim_{n\to\infty} a_n$
Then $L = \frac{1}{2}\left(L + \frac{2}{L}\right)$
Solving: $2L = L + \frac{2}{L} \Rightarrow L = \frac{2}{L} \Rightarrow L^2 = 2 \Rightarrow L = \sqrt{2}$

$\lim_{n\to\infty} a_n = \sqrt{2}$

Question 9

Determine whether the sequence is divergent or convergent. If it is convergent, evaluate its limit.

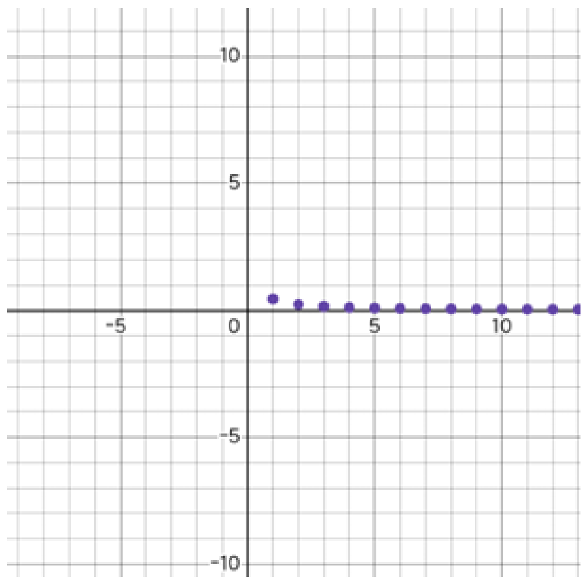
$$\lim_{n\to\infty} (-1)^n \sin(17/n)$$

Answer: 0

Explanation: Since $\sin(17/n) \to 0$ as $n \to \infty$ and $|(-1)^n \sin(17/n)| \le |\sin(17/n)| \to 0$, the sequence converges to 0 by the squeeze theorem.

Question 10

Consider the sequence $\{a_n\} = \left\{ \frac{4n+1}{4n} - \frac{4n}{4n+1} \right\}$. Graph this sequence and use your graph to help you answer the following questions.



Part 1: Boundedness

a. Bounded above by: $a_1 = \frac{5}{4} - \frac{4}{5} = \frac{9}{20}$
b. Bounded below by: 0
c. The sequence is: B. bounded above, C. bounded below, D. bounded

Part 2: Monotonicity

The sequence is: C. decreasing

Part 3: Convergence

a. The sequence is: convergent
b. Limit: 0

Part 4: Conceptual questions

a. The sequence $\left\{ \dfrac{(-1)^n(10n^2+1)}{n^2+n} \right\}$ is: A. divergent, C. not monotonic, E. bounded

For n is even the sequence becomes $\left\{ \dfrac{10n^2+1}{n^2+n} \right\}$ and $\dfrac{10n^2+1}{n^2+n} \leq \dfrac{10n^2+1}{n^2+1} < \dfrac{10n^2+10}{n^2+1} \leq 10$.

For n is odd the sequence becomes $\left\{ -\dfrac{10n^2+1}{n^2+n} \right\}$ and $-\dfrac{10n^2+1}{n^2+n} \geq -\dfrac{10n^2+1}{n^2+1} > -\dfrac{10n^2+10}{n^2+1} \geq -10$. Thus the sequence is bounded above by 10 and bounded below by -10.

Therefore, the sequence is bounded but not monotonic because it changes sign.

For even $n = 2k$, $\dfrac{10n^2+1}{n^2+n} = \dfrac{10+1/n^2}{1+1/n}$, we have

$$\lim_{k\to\infty} a_{2k} = \lim_{n\to\infty} \frac{10+1/n^2}{1+1/n} = \frac{\lim_{n\to\infty} 10+1/n^2}{\lim_{n\to\infty} 1+1/n} = 10,$$

while for odd $n = 2k-1$, $-\dfrac{10n^2+1}{n^2+n} = -\dfrac{10+1/n^2}{1+1/n}$, we have

$$\lim_{k\to\infty} a_{2k-1} = \lim_{n\to\infty} -\frac{10+1/n^2}{1+1/n} = -\frac{\lim_{n\to\infty} 10+1/n^2}{\lim_{n\to\infty} 1+1/n} = -10,$$

The limits of even subsequence and odd subsequence do not match, therefore the sequence is divergent.

b.The sequence $\left\{ \dfrac{10n^3+1}{n^2+n} \right\}$ is: B. divergent, E. monotonic, F. unbounded

We denote the sequence by $a_n = \dfrac{10n^3+1}{n^2+n}$. Then for arbitrary n, we have

$$\begin{aligned} a_{n+1} - a_n &= \frac{10(n+1)^3+1}{(n+1)^2+(n+1)} - \frac{10n^3+1}{n^2+n} \\ &= \frac{10(n+1)^3+1}{(n+2)(n+1)} - \frac{10n^3+1}{(n+1)n} \\ &= \frac{[10(n+1)^3+1]n - (10n^3+1)(n+2)}{(n+2)(n+1)n} \\ &= \frac{10n^3+30n^2+10n-2}{(n+2)(n+1)n} \end{aligned}$$

The numerator $10n^3+30n^2+10n-2 > 10n-2 \geq 8 > 0$ for $n \geq 1$, so $a_{n+1} - a_n > 0$ for arbitrary $n \geq 1$, $n \in \mathbb{N}$, hence the sequence is monotonic increasing.

Note that the following inequality holds for $n \geq 1$:

$$\frac{10n^3+1}{n^2+n} > \frac{10n^3}{n^2+n} \geq \frac{10n^3}{n^2+n^2} = 5n$$

so the sequence is unbounded, hence it's divergent.

c. If a sequence is bounded, it may or may not converge.
d. If a sequence is monotonic, it may or may not converge.
e. If a sequence is bounded and monotonic, it must converge.