Mathematical Induction:

$P(n)$ $\qquad a_n \geq 3$ (1) / $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

i) $P(1)$ is true

$a_1 \geq 3$ $\qquad 1 = \frac{1\cdot(1+1)}{2}$

ii): Assume $P(n)$ is true !

(1): $a_n \geq 3$ $\qquad$ (2): $1+2+\cdots n = \frac{n(n+1)}{2}$

want prove: $P(n+1)$ is true

Conclude: $P(n)$ is true for any positive integer n.

$P(\infty)$

$P(n)$ true, $\forall n$ $\Longrightarrow$ $P(\omega)$ is true

$\frac{1}{n} > 0$ $\qquad$ $0 > 0$

Exercise 1. (Level 1)

Prove that $1 \times 2 + 2 \times 3 + 3 \times 4 + \cdots + n \times (n+1) = \frac{n(n+1)(n+2)}{3}$ for all natural numbers n.

Proof by induction: $P(n)$ is

$$1 \times 2 + 2 \times 3 + \cdots + n \times (n+1) = \frac{n(n+1)(n+2)}{3}$$

i) $P(1)$: $1 \times 2 \overset{\checkmark}{=} \frac{1 \times (1+1) \times (1+2)}{3}$

ii) Assume $P(n)$ is true

$P(n+1)$:

$$1\times 2 + \cdots + n\times(n+1) + (n+1)\times(n+2)$$

$$= \frac{(n+1)(n+2)(n+3)}{3}$$

Apply the hypothesis:

$$\left(1\times 2 + \cdots + n\times(n+1)\right) + (n+1)\times(n+2)$$

$$= \frac{n(n+1)(n+2)}{3} + (n+1)\times(n+2) \qquad (n+3)(n+1)(n+2)$$

$$//$$

$$= \frac{1}{3}\left(n(n+1)(n+2) + 3(n+1)(n+2)\right)$$

Exercise 2. (Level 1)
Prove that $8^n - 3^n$ is divisible by 5 for all natural numbers n.

$$8^n - 3^n = 5 \times \square \longrightarrow \text{integer}$$

Proof by induction: Let $P(n)$ =

"$8^n - 3^n$ is divisible by 5"

i). $P(1)$: $8^1 - 3^1 = 5 \times 1$

ii) : Assume $P(n)$ is true :

There is an integer M such that:

$$8^n - 3^n = 5 \cdot M$$

$P(n+1)$: $$8^{n+1} - 3^{n+1} = \underline{8 \cdot} 8^n - \underline{3 \cdot} 3^n$$

$$= 8 \cdot 8^n - 8 \cdot 3^n + 8 \cdot 3^n - 3 \cdot 3^n$$

$$= 8 \cdot (8^n - 3^n) + 5 \cdot 3^n$$

$$= 8 \cdot 5 \cdot M + 5 \cdot 3^n$$

$$= 5(8M + 3^n)$$ $\square$

Exercise 3. (Level 1)
A sequence $\{a_n\}$ is defined by

$$a_1 = 3 \text{ and } a_{n+1} = \sqrt{a_n + 5} \text{ for } n \geq 1.$$

Show that $a_n \geq a_{n+1}$ for all natural numbers n.

Proof by induction : $P(n)$: " $a_n \geq a_{n+1}$ "

i) $P(1)$: $a_1 \geq a_2$ $\quad a_2 = \sqrt{a_1 + 5}$

$$= \sqrt{3+5} = \sqrt{8} < 3$$

ii) Assume $P(n)$ is true:

$$a_n \geq a_{n+1}$$

$$\Downarrow$$

$P(n+1)$: $a_{n+1} \geq a_{(n+1)+1} = a_{n+2}$

$$a_{n+2} = \sqrt{a_{n+1}+5} \leq \sqrt{a_n+5} = a_{n+1} \qquad \boxed{\checkmark}$$

$$a_n + 5 \geq a_{n+1} + 5$$

$$\Rightarrow \sqrt{a_n+5} \geq \sqrt{a_{n+1}+5}$$

Exercise 4. (Level 1)
A sequence $\{a_n\}$ is defined by

$$a_1 = 4 \text{ and } a_{n+1} = \boxed{\frac{6(a_n^2+1)}{a_n^2+11}} \text{ for } n \geq 1.$$

(a) Show that $a_n > 3$ for all natural numbers n.

(b) Show that $a_n \geq a_{n+1}$ for all natural numbers n.

(a): Proof by induction: $P(n)$: " $a_n > 3$ "

i) $P(1)$: $a_1 > 3$ ✓

ii) Assume $P(n)$ is true: $a_n > 3$

Want to prove: $P(n+1)$ is true $a_{n+1} > 3$

$$a_{n+1} - 3 = \frac{6(a_n^2+1)}{a_n^2+11} - 3$$

$$= \frac{6(a_n^2+1) - 3(a_n^2+11)}{a_n^2+11}$$

$$= \frac{3a_n^2 - 27}{a_n^2+11} > 0$$

$$a_n > 3 \Rightarrow a_n^2 > 9 \Rightarrow 3a_n^2 > 27$$

(b): Proof by induction:

$P(n)$: $a_n \geq a_{n+1}$

i) : $P(1)$ is true: $a_1 \geq a_2$

$$a_2 = \frac{6 \times (a_1^2+1)}{a_1^2+11} = \frac{6 \times (16+1)}{16+11}$$

$$= \frac{6 \times 17}{27}$$

ii) Assume: $P(n)$ Is true:

$$a_n \geq a_{n+1}$$

Want to prove: $a_{n+1} \geq a_{n+2}$

$$a_{n+1} - a_{n+2} = \frac{6(a_n^2+1)}{a_n^2+11} - \frac{6(a_{n+1}^2+1)}{a_{n+1}^2+11} \geq 0$$

$$\frac{6(a_n^2+11)-60}{a_n^2+11} = 6 - \frac{60}{a_n^2+11}$$

$$a_{n+1} - a_{n+2} = \left(6 - \frac{60}{a_n^2+11}\right) - \left(6 - \frac{60}{a_{n+1}^2+11}\right)$$

$$= 60\left(\frac{1}{a_{n+1}^2+11} - \frac{1}{a_n^2+11}\right)$$

$$\frac{1}{a_{n+1}^2+1} - \frac{1}{a_n^2+1} = \frac{1}{\boxed{}}\left(a_n^2+1-(a_{n+1}^2+1)\right)$$

$$= \frac{1}{\boxed{}}\underbrace{\left(a_n^2 - a_{n+1}^2\right)}_{\geq 0} \geq 0$$

$(a_n^2+1)\cdot(a_{n+1}^2+1)$

$$P(1) \Rightarrow P(2) \Rightarrow P(3) \cdots \Rightarrow P(n)$$

i) $P(1)$ ii) $P(n) \Rightarrow P(n+1)$

i) $P(2)$ ii) $P(n) \Rightarrow P(n+1)$

$$P(2) \Rightarrow \cdots \Rightarrow P(n)$$

i) $P(k)$ ii) $P(n) \Rightarrow P(n+1)$

$$P(k) \Rightarrow P(k+1) \Rightarrow \cdots \Rightarrow P(n)$$

2) $P(1) \Rightarrow P(2) \Rightarrow \cdots \Rightarrow P(n)$

i) $P(1)$ is true

ii) $P(n) \Rightarrow P(n+1)$

$P(2)$ is true

$P(n-1), P(n) \Rightarrow P(n+1)$

$P(3)$

$P(n-2)$

$P(3) \Leftarrow P(1), P(2)$

$\vdots$

$P(4) \Leftarrow P(2), P(3)$

$P(k)$ is true

for any $1 \leq k \leq n$

i) $P(1)$ ✓

ii) $P(k)$ $k \leq n$

$P(n+1)$

$$\begin{matrix} P(1) \\ P(2) \end{matrix} \Rightarrow \begin{matrix} P(3) \\ \\ P(2) \end{matrix} \Rightarrow \begin{matrix} P(4) \\ P(3) \end{matrix} \Rightarrow P(5)$$

$\ldots$