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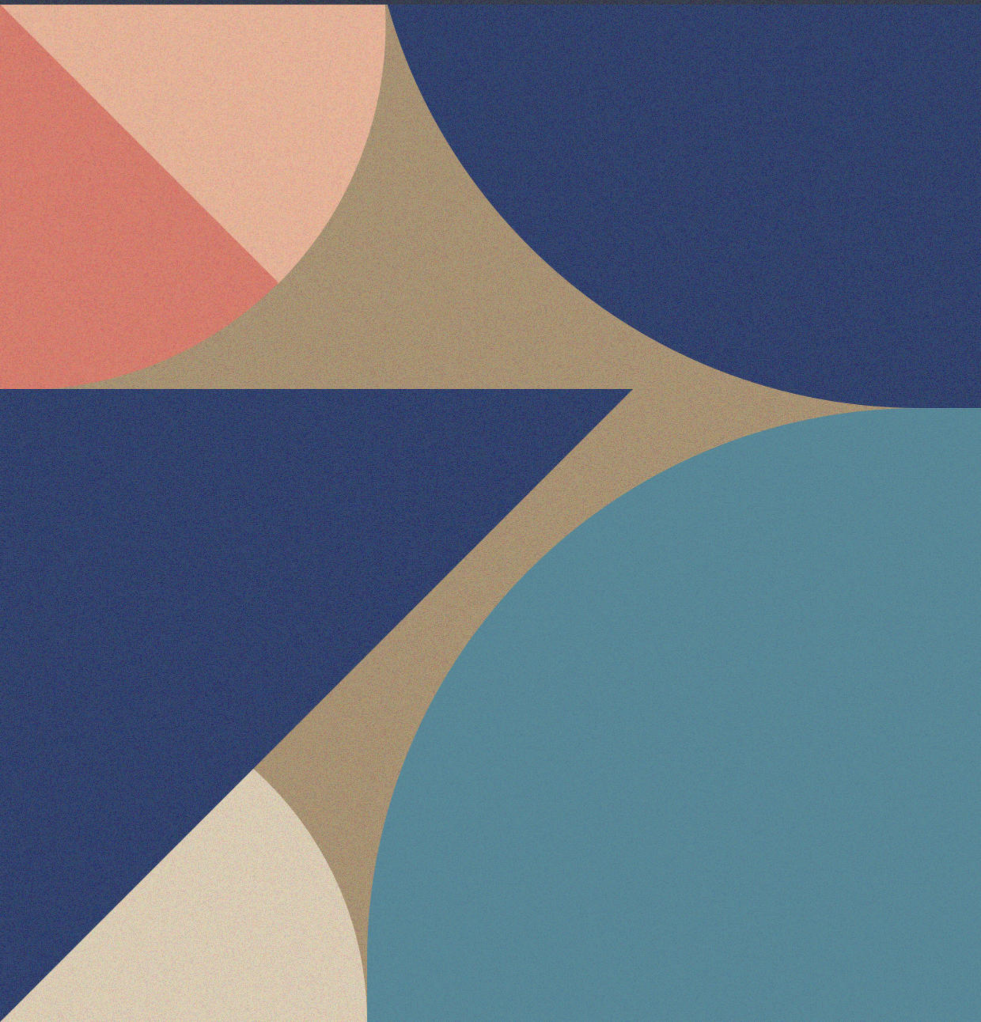


Math1010F
Sep. 11, 2025

Ex 1.1.# Let $a_1 = 1$ and $a_n = \frac{12(a_{n-1}+1)}{a_{n-1}+13}$ $n \geq 1$

(a) Prove that $a_n \leq 3$

Proof: Statement $P(n)$: $a_n \leq 3$ is about natural number

I Base step $n=1$ $a_1 = 1 \leq 3$ ✓

II Induction step: Assume $a_n \leq 3$

$$a_{n+1} - 3 = \frac{12(a_n+1)}{a_n+13} - 3 = \frac{9a_n - 27}{a_n+13} = \frac{9(a_n-3)}{a_n+13} \leq 0$$

that is $a_{n+1} \leq 3$ ✓

By Mathematical Induction $a_n \leq 3$ □

Method II . we have $a_{n+1} = \frac{12(a_n+1)}{a_n+13} = \frac{12(a_n+1)}{a_n+1+12}$

so $\frac{1}{a_{n+1}} = \frac{1}{12} + \frac{1}{a_n+1}$. Since $a_n \leq 3$

$\Rightarrow a_n + 1 \leq 4$ that is $\frac{1}{a_n+1} \geq \frac{1}{4}$

Hence $\frac{1}{a_{n+1}} \geq \frac{1}{12} + \frac{1}{4} = \frac{1}{3}$, that is. $a_{n+1} \leq 3$ ✓

Ex 1.1.# Let $a_1=1$ and $a_n = \frac{12(a_{n-1}+1)}{a_{n-1}+13}$ $n \geq 1$

(a) Prove that $a_n \leq 3$

(b) Prove that $\{a_n\}$ converges and find its limit

by (a). (Target)

$\{a_n\}\uparrow$ + $a_n \leq 3$ (upper bound) $\Rightarrow$ $\{a_n\}$ converges

Prof: Statement $P(n)$: $a_n \leq a_{n+1}$

1. Base step: $a_1 = 1$ $a_2 = \frac{12+12}{1+13} = \frac{12}{7}$ ✓

II. Induction step: Assume that $a_n \leq a_{n+1}$

$$a_{n+2} - a_{n+1} = \frac{12(a_{n+1}+1)}{a_{n+1}+13} - \frac{12(a_n+1)}{a_n+13}$$

$$= \frac{12[(a_{n+1}+1)(a_n+13) - (a_n+1)(a_{n+1}+13)]}{(a_{n+1}+13)(a_n+13)} = \frac{12 \times 12 \overbrace{(a_{n+1} - a_n)}^{\geq 0}}{\underbrace{(a_{n+1}+13)}_{\geq 0}\underbrace{(a_n+13)}_{\geq 0}} \geq 0$$

that is $a_{n+2} \geq a_{n+1}$ ✓

By mathematical Induction $\{a_n\}\uparrow$ □

We know that

$$a_n = \frac{12a_{n-1}+12}{a_{n-1}+13}$$

$$\lim_{n\to+\infty} a_n = \frac{12\lim_{n\to+\infty} a_{n-1}+12}{\lim_{n\to+\infty} a_{n-1}+13}$$

Let $\lim_{n\to+\infty} a_n = A$

so $$A = \frac{12A+12}{A+13}$$

that is $A^2+A-12=0$

so $A=3$ or $A=-4$ (rejected)

Hence $\lim_{n\to+\infty} a_n = 3$. $\square$

Check! $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} a_{n+k} = \lim_{n\to+\infty} a_{2n} = \lim_{n\to+\infty} a_{2n-1}$ (for given k)

In general, for infinite set $K \subset \mathbb{N}_+$ and $K=\{n_1<n_2<\cdots<n_k<\cdots\}$ we have $\lim_{n\to+\infty} a_n = \lim_{k\to+\infty} a_{n_k}$

Method II about Ex 1.1# (b) $a_n \le a_{n+1}$

Proof by Contradiction 反證法 ☆

Ex 1.1# Let $a_1 = 1$ and $a_n = \frac{12(a_{n-1}+1)}{a_{n-1}+13}$ $n \ge 1$

(a) Prove that $a_n \le 3$

(b) Prove that $\{a_n\}$ converges and find its limit

Proof: Assume $a_n \nleq a_{n+1}$ $n > 1$

that is $a_n \le a_{n-1} \le \cdots \le a_1 = 1$ $(*)$

Now, since that $a_{n+1} = \frac{12a_n + 12}{a_n + 13}$,

it holds that $a_n \ge a_{n+1} = \frac{12a_n + 12}{a_n + 13}$.

Hence $a_n^2 + a_n - 12 \ge 0$

$\Leftrightarrow (a_n + 4)(a_n - 3) \ge 0$

that is $\begin{cases} a_n \ge 3 \\ a_n \le -4 \end{cases}$

Rejected $a_n \ge 3$: By $(*)$, $a_n \le 1$

Rejected $a_n \le -4$: it is easy to proof that $a_n > 0$

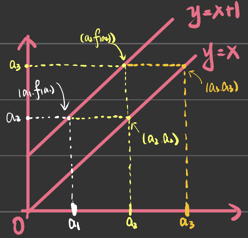
No solution! This is a contradiction! Hence $a_n > a_{n+1}$

Question for Ex 11# 1b)

Here, given that Monotone Convergence Theorem is to be applied, we guess that $\{a_n\}$ is monotonic increasing. If we don't know this matter, what methods do we have to judge the changing trend of $\{a_n\}$ (even in non-monotonic cases)?

★ Hint: Let's consider $a_1 = 1$, $a_{n+1} = a_n + 1$ and $f(x) = x+1$

that is, $a_{n+1} = f(a_n)$ $n \geq 1$.



Here, we can observe the changing trend of $\{a_n\}$ on the number axis.

Ex 1.2#

$$\begin{cases} x_1 = 2 \\ x_{n+1} = \dfrac{x_n y_n (x_n+y_n)}{x_n^2+y_n^2} \end{cases} \qquad \begin{cases} y_1 = 8 \\ y_{n+1} = \dfrac{x_n^2+y_n^2}{x_n+y_n} \end{cases}$$

(a) Prove that $x_{n+1} - y_{n+1} = \dfrac{-(x_n^3-y_n^3)(x_n-y_n)}{(x_n+y_n)(x_n^2+y_n^2)}$

Proof:
$$\begin{aligned} x_{n+1} - y_{n+1} &= \frac{x_n y_n (x_n+y_n)}{x_n^2+y_n^2} - \frac{x_n^2+y_n^2}{x_n+y_n} \\ &= \frac{x_n y_n (x_n+y_n)^2 - (x_n^2+y_n^2)^2}{(x_n^2+y_n^2)(x_n+y_n)} \\ &= \frac{x_n^3 y_n + \cancel{2x_n^2 y_n^2} + x_n y_n^3 - x_n^4 - \cancel{2x_n^2 y_n^2} - y_n^4}{(x_n^2+y_n^2)(x_n+y_n)} \\ &= \frac{x_n^3(y_n-x_n) + y_n^3(x_n-y_n)}{(x_n^2+y_n^2)(x_n+y_n)} \\ &= \frac{-(x_n^3-y_n^3)(x_n-y_n)}{(x_n^2+y_n^2)(x_n+y_n)}. \end{aligned}$$

$\square$

Ex 1.2.# $\begin{cases} x_1 = 2 \\ x_{n+1} = \dfrac{x_n y_n (x_n + y_n)}{x_n^2 + y_n^2} \geq 0 \end{cases}$ $\begin{cases} y_1 = 8 \\ y_{n+1} = \dfrac{x_n^2 + y_n^2}{x_n + y_n} \geq 0 \end{cases}$

(a) Prove that $\boxed{x_{n+1} - y_{n+1} = \dfrac{-(x_n^3 - y_n^3)(x_n - y_n)}{(x_n + y_n)(x_n^2 + y_n^2)}}$

(b) Show $0 \leq x_n \leq y_n$. and prove $\{x_n\}\uparrow$, $\{y_n\}\downarrow$

Prof: Statement $P(n)$: $0 \leq x_n \leq y_n$.

I Base step: $x_1 = 2$ $y_1 = 8$ $x_1 < y_1$ ✓

II Induction step: Assume $x_n \leq y_n$

$$y_{n+1} - x_{n+1} = \frac{(x_n^3 - y_n^3)(x_n - y_n)}{(x_n + y_n)(x_n^2 + y_n^2)} \geq 0$$

we know $x_n \leq y_n$ (assume)

so $x_n - y_n \leq 0$ $x_n^3 - y_n^3 \leq 0$

that is $x_{n+1} \leq y_{n+1}$ ✓

By mathematical induction, $x_n \leq y_n$ $\forall n$

Prove $\{x_n\}\uparrow$, $\{y_n\}\downarrow$ ← equivalent 单调

$$x_n \le x_{n+1} \qquad\qquad y_n \ge y_{n+1}$$

Let $x_{n+1}-x_n = \dfrac{x_n^2 y_n + x_n y_n^2}{x_n^2+y_n^2} - x_n = \dfrac{-x_n^2\overbrace{(x_n-y_n)}^{\le 0}}{x_n^2+y_n^2} \ge 0$

that is $\{x_n\}\uparrow$

Let $y_{n+1}-y_n = \dfrac{x_n^2+y_n^2}{x_n+y_n} - y_n = \dfrac{x_n\overbrace{(x_n-y_n)}^{\le 0}}{x_n+y_n} \le 0$

that is $\{y_n\}\downarrow$

$\square$

Ex 1.2#

$$\begin{cases} x_1 = 2 \\ x_{n+1} = \dfrac{x_n y_n (x_n+y_n)}{x_n^2+y_n^2} \end{cases} \qquad \begin{cases} y_1 = 8 \\ y_{n+1} = \dfrac{x_n^2+y_n^2}{x_n+y_n} \end{cases}$$

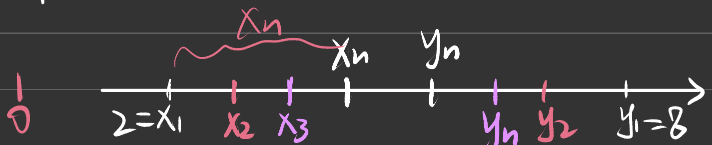
(a) Prove that $x_{n+1}-y_{n+1} = \dfrac{-(x_n^3-y_n^3)(x_n-y_n)}{(x_n+y_n)(x_n^2+y_n^2)}$

(b) Show $\boxed{0 \le x_n \le y_n}$. and prove $\{x_n\}\uparrow$, $\{y_n\}\downarrow$

(c) Prove $\{x_n\}.\{y_n\}$ converge and $\lim\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} y_n$

$\{x_n\}\uparrow$ increasing + upper bound. $\Rightarrow$ $\{x_n\}$ converges

Prof: We draw $\{x_n\}$ $\{y_n\}$ in number axis.



Find $x_2.y_2$ $\quad x_2 \ge x_1 \quad y_2 \le y_1 \quad x_2 \le y_2$

Find $x_3.y_3$ $\quad x_3 \ge x_2 \quad y_3 \le y_2 \quad x_3 \le y_3$ ......

For every n

$x_n \le y_n \le y_1 = 8$ $\quad \{x_n\}$ has a

upper bound that is $\{a_n\}$ converges

For every n

$$2 = x_1 \le x_n \le y_n$$

that is $\{y_n\}$ has a lower bound.

Hence $\{y_n\}$ converges.

Prove $\lim\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} y_n$

Let $\lim\limits_{n\to+\infty} x_n = X$ we have

$\lim\limits_{n\to+\infty} y_n = Y$

$$x_{n+1} = \frac{x_n^2 y_n + x_n y_n^2}{x_n^2 + y_n^2}$$

$$\Big\downarrow \lim_{n\to+\infty}$$

$$X = \frac{X^2 Y + X Y^2}{X^2 + Y^2}$$

so. $X^3 + \cancel{XY^2} = X^2 Y + \cancel{XY^2}$

$$\Rightarrow X^2(X-Y) = 0 \Rightarrow \begin{cases} X = 0 \text{ (rejected)} \\ X = Y \end{cases}$$

$\square$

Ex 1.2# $\begin{cases} x_1 = 2 \\ x_{n+1} = \dfrac{x_n y_n (x_n+y_n)}{x_n^2+y_n^2} \end{cases}$ $\begin{cases} y_1 = 8 \\ y_{n+1} = \dfrac{x_n^2+y_n^2}{x_n+y_n} \end{cases}$

(a) Prove that $x_{n+1} - y_{n+1} = \dfrac{-(x_n^3-y_n^3)(x_n-y_n)}{(x_n+y_n)(x_n^2+y_n^2)}$

(b) Show $0 \le x_n \le y_n$. and prove $\{x_n\}\uparrow$, $\{y_n\}\downarrow$

(c) Prove $\{x_n\}$. $\{y_n\}$ converge and $\lim\limits_{n\to+\infty} x_n = \lim\limits_{n\to+\infty} y_n$

(d) Prove $x_n y_n$ is constant and find $\lim\limits_{n\to+\infty} x_n$

Proof: $x_{n+1} y_{n+1} = \overset{x_{n+1}}{\dfrac{x_n y_n \cancel{(x_n+y_n)}}{\cancel{x_n^2+y_n^2}}} \times \overset{y_{n+1}}{\dfrac{\cancel{x_n^2+y_n^2}}{\cancel{x_n+y_n}}} = x_n y_n$

that is for every n we have

$$x_n y_n = x_{n-1} y_{n-1} = \cdots\cdots = x_2 y_2 = x_1 y_1 = 2 \times 8 = 16$$

constant

$x_n y_n = 16$ let $\lim\limits_{n\to+\infty}$, $XY = 16$

By (c) $X = Y$ we have $X = Y = 4$

X or $Y = -4$ (rejected)

□

Ex 1.3# Use Sandwich Theorem prove that

$$\lim_{n\to+\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right]=1.$$

Proof Since that $\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+k}} \geq \frac{1}{\sqrt{n^2+n}}$ $(\forall k=1,2,\cdots n)$

$\sqrt{n^2+1} \leq \sqrt{n^2+k} \leq \sqrt{n^2+n}$

$$\text{sum}\begin{cases} \frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+n}} \\ \frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+2}} \geq \frac{1}{\sqrt{n^2+n}} \\ \frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+n}} \geq \frac{1}{\sqrt{n^2+n}} \end{cases}$$

$\Rightarrow$ that is

$$\sum_{k=1}^{n}\frac{1}{\sqrt{n^2+1}} \geq \sum_{k=1}^{n}\frac{1}{\sqrt{n^2+k}} \geq \sum_{k=1}^{n}\frac{1}{\sqrt{n^2+n}}$$

$$\underbrace{\frac{n}{\sqrt{n^2+1}}}_{=C_n} \geq \underbrace{\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}}_{=b_n} \geq \underbrace{\frac{n}{\sqrt{n^2+n}}}_{=a_n}$$

Since $\lim_{n\to+\infty} C_n = \lim_{n\to+\infty}\frac{n}{\sqrt{n^2+1}} = \lim_{n\to+\infty}\sqrt{\frac{1}{1+\frac{1}{n^2}}} = 1$, $\lim_{n\to+\infty} a_n = 1$,

by Sandwich Theorem, we have $\{b_n\}$ converges and $\lim_{n\to+\infty} b_n = 1$ $\square$