Title ____________________

Ex 1.1# Find the critical point and determi neif the function is increasing or decreasing on the given intervals. Consider

$$y = -x^2 + 8x - 9.$$

(a) The critical point c is ?.

(b) Determine the monotonicity of y on $(-\infty, c)$ and $(c, +\infty)$.

Proof. (a) The derivative of $y = -x^2 + 8x - 9$ is

$$y' = -2x + 8.$$

Letting $y' = 0$, we obtain that $x = 4$. Hence, $c = 4$ is the critical point.

(b) Noting that

$$\begin{cases} y' = -2x + 8 < 0, & \text{when } x > 4; \\ y' = -2x + 8 > 0, & \text{when } x < 4. \end{cases}$$

Hence, y is increasing on $(-\infty, 4)$ and decreasing on $(4, +\infty)$. $\square$

Ex 1.2# Find the critical point and determine if the function is increasing or decreasing, and apply the First Derivative Test to the critical point. Consider

$$f(x) = 2x - 2\log(8x), \quad x \in \mathbb{R}_+.$$

1a) What is the critical point.

1b) Is f a maximum and minimum at the critical point.

1c) The open interval on the left of the critical point is? On this intervals, f is (increasing | decreasing) while f' is (positive \ negative)?

1d) The open interval on the right of the critical point is? On this intervals, f is (increasing | decreasing) while f' is (positive \ negative)?

Proof. 1(a) Note that $f'(x) = 2 - \frac{2}{x}$. Letting $f'(x) = 0$, we obtain that $x = 1$. Hence, the critical point is $x = 1$.

1(b) Let us compute the second derivative $f''(x) = \frac{2}{x^2}$. Since $f''(x) > 0$ for all $x > 0$, it follows that $f'(x)$ is increasing. Hence, $f'(x)$ is negative on $(0, 1)$ and positive on $(1, +\infty)$, which implies that $f(x)$ is minimum at the critical point $x = 1$.

1(c) As shown in 1(b), the interval on the left of the critical point is $(0, 1)$. Then, $f(x)$ is decreasing and $f'(x)$ is negative on this interval.

1(d) As shown in 1(b), the interval on the right of the critical point is $(1, +\infty)$. Then, $f(x)$ is decreasing and $f'(x)$ is negative on this interval. $\square$

Ex 1.3[4] Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = x \log(1+|x|), \quad \text{for any } x \in \mathbb{R}.$$

(a) Express the explicit formula of the function f.

(b) Show that f is continuous at 0.

(c) Compute $f'(x)$ for each $x \in \mathbb{R}\setminus\{0\}$, and also compute $\lim_{x\to 0^+} f'(x)$ and $\lim_{x\to 0^-} f'(x)$.

Proof. (a) Since $f(x) = x\log(1+|x|)$, then we have that

$$f(x) = \begin{cases} x\log(1-x), & \text{if } x<0; \\ x\log(1+x), & \text{if } x\geq 0. \end{cases}$$

(b) Noting that $f(0)=0$,

$$\begin{cases} \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} [x\log(1+x)] = 0; \\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} [x\log(1-x)] = 0, \end{cases}$$

we have that f is continuous at $x=0$.

(c) For each $x \in \mathbb{R}\setminus\{0\}$, we have that

$$f'(x) = \begin{cases} \log(1-x) - \frac{x}{1-x}, & \text{if } x<0; \\ \log(1+x) + \frac{x}{1+x}, & \text{if } x\geq 0, \end{cases}$$

then, it implies that

$$\begin{cases} \lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} [\log(1-x) - \frac{x}{1-x}] = 0; \\ \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} [\log(1+x) + \frac{x}{1+x}] = 0. \end{cases}$$

$\square$

Ex 1.3[#] Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = x \log(1+|x|), \quad \text{for any } x \in \mathbb{R}.$$

(d) Is f differentiable at $f'(0)$? If yes, also find the value of $f'(0)$.

(e) Is f continuously differentiable at $x=0$?

(f) Is f twice-differentiable at $x=0$?

Proof. (d) Since f is continuous at $x=0$, and both limit $\lim_{x\to 0^-} f'(x)$, $\lim_{x\to 0^+} f'(x)$ exist and are equal to each other, f is differentiable at $x=0$. Moreover, $f'(0)=0$.

(e) Since $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x) = f'(0)$, f' is continuous at $x=0$. Hence, f is continuously differentiable at $x=0$.[†]

(f) For each $x \in \mathbb{R}\setminus\{0\}$, we have that

$$f''(x) = \begin{cases} \dfrac{x-2}{(1-x)^2}, & \text{if } x<0; \\ \dfrac{x+2}{(1+x)^2}, & \text{if } x>0. \end{cases}$$

then, it holds that

$$\begin{cases} \lim_{x\to 0^-} f''(x) = \lim_{x\to 0^-} \dfrac{x-2}{(1-x)^2} = -2; \\ \lim_{x\to 0^+} f''(x) = \lim_{x\to 0^+} \dfrac{x+2}{(1+x)^2} = 2. \end{cases}$$

Since $\lim_{x\to 0^-} f''(x)$, $\lim_{x\to 0^+} f''(x)$ are not equal to each other. Then, f' is not differentiable at 0. Hence, f is not twice-differentiable at $x=0$. $\square$

Note[†]: If f is differentiable, it is not always true that f' is continuous.

Hint: $f(x) = x^2 \sin\frac{1}{x}$ when $x \neq 0$; $f(x)=0$, when $x=0$.

Ex 1.4# Let $a, b \in \mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \frac{1}{x}(2 + a\sqrt{1-x}), & \text{if } x < 0; \\ 1 + b\tan\frac{x}{8}, & \text{if } x \geq 0. \end{cases}$$

It is given that f is continuous at 0.

(a) Determine the value of a.

Proof. (a) Since f is continuous at 0, we have

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0),$$

where $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (1 + b\tan\frac{x}{8}) = 1$. Because $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x}(2 + a\sqrt{1+x})$ exists, then numerator at $x = 0$ must be 0.

(If $\lim_{x \to 0^-}(2 + a\sqrt{1+x}) \neq 0$, it is obvious that $\lim_{x \to 0^-} \frac{1}{x}(2 + a\sqrt{1+x}) \to +\infty$)

Hence, $\lim_{x \to 0^+}(2 + a\sqrt{1+x}) = 0$, that is, $a = -2$. $\square$

Ex 1.4# Let $a, b \in \mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \frac{1}{x}(2 + a\sqrt{1-x}), & \text{if } x < 0; \\ 1 + b\tan\frac{x}{8}, & \text{if } x \geq 0. \end{cases}$$

It is given that f is continuous at 0.

(a) Determine the value of a.

(b) It is further assume that f is differentiable at 0. Determine the value of b.

Proof. (b) Since f is differentiable at 0, $f'_+(0)$ and $f'_-(0)$ exist and are equal. Noting that

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^-} \frac{1}{h}\left(\frac{2-2\sqrt{1-h}}{h} - 1\right) \qquad (f(0)=1) \\
&= \lim_{h\to 0^-} \frac{(2-h) - 2\sqrt{1-h}}{h^2} \\
&= \lim_{h\to 0^-} \frac{(2-h) - 2\sqrt{1-h}}{h^2} \cdot \frac{(2-h)+2\sqrt{1-h}}{(2-h)+2\sqrt{1-h}} \\
&= \lim_{h\to 0^-} \frac{(2-h)^2 - 4(1-h)}{h^2[(2-h)+2\sqrt{1-h}]} \\
&= \lim_{h\to 0^-} \frac{h^2}{h^2[(2-h)+2\sqrt{1-h}]} = \frac{1}{4}
\end{aligned}$$

and

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} &= \lim_{h\to 0^+} \frac{1 + b\tan\frac{h}{8} - 1}{h} \\
&= \lim_{t\to 0^+} \frac{b\tan t}{8t} \qquad (\text{Here, we let } h = 8t) \\
&= \frac{b}{8} \lim_{t\to 0^+} \frac{\sin t}{t} \cdot \lim_{t\to 0^+} \frac{1}{\cos t} = \frac{b}{8},
\end{aligned}$$

it holds that $\frac{b}{8} = \frac{1}{4}$. Hence, $b = 4$. □

Ex 1.5* Let $f, g: \mathbb{R} \to \mathbb{R}$ be two functions such that

- $g(x+y) = g(x)f(y) + f(x)g(y)$ for all $x, y \in \mathbb{R}$;
- $f(0)=1$, $f'(0)=0$, $g(0)=0$ and $g'(0)=1$.

Show that $g'(x) = f(x)$ for all $x \in \mathbb{R}$.

Proof. Let $x \in \mathbb{R}$. By First Principle, we have that

$$\begin{aligned}
\lim_{h\to 0} \frac{g(x+h)-g(x)}{h} &= \lim_{h\to 0} \frac{g(x)f(h)+f(x)g(h)-g(x)}{h} \\
&= \lim_{h\to 0} \left[g(x)\cdot \frac{f(h)-1}{h} + f(x)\cdot \frac{g(h)}{h} \right] \quad f(0)=1 \text{ and } g(0)=0 \\
&= g(x) \lim_{h\to 0} \frac{f(h)-f(0)}{h} + f(x) \lim_{h\to 0} \frac{g(h)-g(0)}{h} \\
&= g(x)f'(0) + f(x)g'(0) \quad f'(0)=0 \text{ and } g'(0)=1 \\
&= f(x)
\end{aligned}$$

Hence, $g'(x) = f(x)$, for all $x \in \mathbb{R}$. $\square$

Ex 1.6# Let $k \in \mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f(x) = \begin{cases} \sin x + \cos(2x) + k, & \text{if } x \leq 0; \\ xe^{3x}, & \text{if } x > 0. \end{cases}$$

1(a) Is it given that f is continuous at 0.

A. Determine the value of k. B. Is f differentiable at 0?

C. Determine the explicit formula for the function f'.

D. Is f' continuous at 0?

Proof. 1(a) Since f is continuous at 0, we have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$. Noting that

$$\begin{cases} \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} xe^{3x} = 0; \\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} [\sin x + \cos(2x) + k] = 1+k, \end{cases}$$

so we have $1+k=0$, that is, $k=-1$ (A) and $f(0)=0$.

If f is differentiable at 0, $f'_+(0)$ and $f'_-(0)$ exist and are equal. Since

$$\begin{cases} \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{1}{h}\cdot he^{3h} = \lim_{h\to 0^+} e^{3h} = 1; \\ \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{\sin h + \cos(2h) - 1}{h} = \lim_{h\to 0^-} \frac{\sin h - 2\sin^2 h}{h} = 1, \end{cases}$$

we have that f is differentiable at 0 (B) and $f'(0)=0$. Then, we can determine the explicit formula for f', say. 1(c)

$$f'(x) = \begin{cases} \cos x - 2\sin(2x), & \text{if } x<0; \\ 1, & \text{if } x=0; \\ e^{3x} + 3xe^{3x}, & \text{if } x>0. \end{cases}$$

Noting that $f'(0)=1$, $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} [\cos x - 2\sin(2x)] = 1$ and $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} [\cos x - 2\sin(2x)] = 1$, so f' is continuous at $x=0$ (D). $\square$

Ex 1.6# Let $k \in \mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}$ is the function defined by

$$f(x) = \begin{cases} \sin x + \cos(2x) + k, & \text{if } x \leq 0; \\ xe^{3x}, & \text{if } x > 0. \end{cases}$$

(b) It is instead given that $k = 0$.

A. Compute f' for each non-zero value of x.

B. Compute $\lim_{x\to 0^+} f'(x)$ and $\lim_{x\to 0^-} f'(x)$.

C. Is f differentiable at 0 ?

Proof. (b) Since $k=0$. by (a). we know that f is not continuous at $x=0$. Hence. f is not differentiable at $x=0$ (C).

Next. for each $x = \mathbb{R} \setminus \{0\}$. we have that (A)

$$f'(x) = \begin{cases} \cos x - 2\sin(2x), & \text{if } x < 0; \\ e^{3x} + 3xe^{3x}, & \text{if } x > 0. \end{cases}$$

Then. we can compute (B)

$$\begin{cases} \lim_{x\to 0^+} f'(x) = \lim_{x\to 0^+} [e^{3x} + 3xe^{3x}] = 1; \\ \lim_{x\to 0^-} f'(x) = \lim_{x\to 0^-} [\cos x - 2\sin(2x)] = 1. \end{cases}$$

$\square$

Ex 1.7# Suppose f has an inverse function and its inverse is twice differentiable. Find a closed formula for

$$\frac{d^2}{dx^2} f^{-1}(x).$$

Review: First, let's recall how we calculated $\frac{d}{dx} f^{-1}(x)$. Since $f^{-1}(f(x)) = x$, we take derivative on both sides and use chain rule, it holds that

$$(f^{-1})'(f(x)) \cdot f'(x) = 1. \qquad (*)$$

where we use $\frac{d}{dx} g(f(x)) = f'(x)\, g'(f(x))$. Hence, let $t = f(x)$. $(*)$ can be rewritten as

$$\frac{d}{dt} f^{-1}(t) = (f^{-1})'(t) = [f'(x)]^{-1} = [f'(f^{-1}(t))]^{-1}. \qquad (**)$$

Proof. Let's differentiate $(*)$ again, we obtain

$$\begin{aligned}
0 &= \frac{d}{dx}\left[(f^{-1})'(f(x)) \cdot f'(x) \right] \quad \text{Product Rule} \\
&= \frac{d}{dx}\left[(f^{-1})'(f(x)) \right] f'(x) + (f^{-1})'(f(x)) \cdot f''(x) \quad \text{Chain Rule} \\
&= \left[f'(x) \cdot (f^{-1})''(f(x)) \right] f'(x) + (f^{-1})'(f(x)) \cdot f''(x).
\end{aligned}$$

Thus, combining with $(**)$, we have that

$$(f^{-1})''(f(x)) \cdot (f'(x))^2 = -[f'(x)]^{-1} \cdot f''(x).$$

Similarly, letting $t = f(x)$, it holds that $x = f^{-1}(t)$ and

$$(f^{-1})''(t) = -[f'(f^{-1}(t))]^{-3} \cdot f''(f^{-1}(t)). \qquad \square$$

Alternative method. (Chain Rule)

Proof. From (**), we know that $\frac{d}{dx}f^{-1}(x) = [f'(f^{-1}(x))]^{-1}$, that is

$$\frac{d^2}{dx^2}f^{-1}(x) = \frac{d}{dx}\left(\frac{d}{dx}f^{-1}(x)\right) = \frac{d}{dx}[f'(f^{-1}(x))]^{-1}. \qquad (1)$$

Let $y=u^{-1}$, $u=f'(w)$ and $w=f^{-1}(x)$, we have

$$[f'(f^{-1}(x))]^{-1} = [f'(w)]^{-1} = u^{-1} = y. \qquad (2)$$

Combining (1) and (2), by chain rule, we have that

$$\begin{aligned}
\frac{d^2}{dx^2}f^{-1}(x) &= \frac{dy}{dx} \qquad (y \to u \to w \to x) \\
&= \frac{dy}{du}\cdot\frac{du}{dw}\cdot\frac{dw}{dx} \\
&= -\frac{1}{u^2}\cdot f''(w)\cdot\frac{d}{dx}(f^{-1}(x)) \\
&= -[f'(f^{-1}(x))]^{-2}\cdot f''(f^{-1}(x))\cdot[f'(f^{-1}(x))]^{-1} \\
&= -[f'(f^{-1}(x))]^{-3}\cdot f''(f^{-1}(x)).
\end{aligned}$$

$\square$

Ex 1.8# Consider the function $\tanh: \mathbb{R} \to \mathbb{R}$.

(a) Show, from the definition of injectivity, that tanh is injective on $\mathbb{R}$.

Proof. (a) First, we recall the definition of injectivity: for all $a, b \in X$, $f(a) = f(b)$ implies that $a = b$.

Pick any $x, w \in \mathbb{R}$. Suppose $\tanh x = \tanh w$, then

$$4 \tanh x = 4 \tanh w$$

$$4 \sinh x \cosh w = 4 \sinh w \cosh x$$

$$(e^x - e^{-x})(e^w + e^{-w}) = (e^w - e^{-w})(e^x + e^{-x})$$

$$\cancel{e^{x+w}} + e^{x-w} - e^{-x+w} - \cancel{e^{-x-w}} = \cancel{e^{x+w}} + e^{w-x} - e^{x-w} - \cancel{e^{-x-w}}$$

$$e^{x-w} - e^{w-x} = e^{w-x} - e^{x-w}$$

$$2e^{x-w} = 2e^{w-x}$$

$$x - w = w - x$$

$f(x) = e^x$ is strictly increasing

Hence, $x = w$, that is, tanh is injective on $\mathbb{R}$. $\square$

Alternative Method (Using calculus):

Proof. For any $x \in \mathbb{R}$, we take derivative on the both sides and obtain that

$$\frac{d}{dx} \tanh x = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \qquad \begin{cases} \frac{d}{dx}(\sinh x) = \cosh x \\ \frac{d}{dx}(\cosh x) = \sinh x \end{cases}$$

$$= \frac{1}{4\cosh^2 x}\left[(e^x + e^{-x})^2 - (e^x - e^{-x})^2\right] \geq \frac{1}{\cosh^2 x} > 0.$$

Hence, tanh is strictly increasing on $\mathbb{R}$, that is, tanh is injective. $\square$

Ex 1.8# Consider the function $\tanh: \mathbb{R} \to \mathbb{R}$.

(b) Show that $-1 < \tanh x < 1$ for any $x \in \mathbb{R}$.

(d) What is the image of $\mathbb{R}$ under tanh?

Proof. (b) Suppose $x \in \mathbb{R}$. Noting that

$$\begin{cases} \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} > \dfrac{-e^x - e^{-x}}{e^x + e^{-x}} = -1; \\ \tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} < \dfrac{e^x + e^{-x}}{e^x + e^{-x}} = 1. \end{cases}$$

Hence, $-1 < \tanh x < 1$.

Alternative Method (Use Monotonicity):

Proof. By the alternative method of (a), we know that $\frac{d}{dx}(\tanh x) > 0$, that is, $\tanh x$ is strictly increasing. That is to say, $\lim_{x\to-\infty} \tanh x \leq \tanh x \leq \lim_{x\to+\infty} \tanh x$.

Noting that

$$\begin{cases} \lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to-\infty} \dfrac{e^{2x} - 1}{e^{2x} + 1} = -1; \\ \lim_{x\to+\infty} \tanh x = \lim_{x\to+\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to+\infty} \dfrac{1 - e^{-2x}}{1 + e^{-2x}} = 1. \end{cases}$$

Besides, it is obvious that $\tanh x \neq \pm 1$. (If $\tanh x = 1$, we have $e^x + e^{-x} = e^x - e^{-x}$, that is, $2e^{-x} = 0$, which is a contradiction) Hence, $-1 < \tanh x < 1$.

Proof. (d) By the statement of (b), the image of $\mathbb{R}$ under $\tanh x$ is $(-1, 1)$. $\square$

Ex 1.8# Consider the function $\tanh: \mathbb{R} \to \mathbb{R}$.

1c) Find, for every value $y \in (-1, 1)$, a solution of the equation $y = \tanh x$ with unknown x.

1e) Hence, or otherwise, write down the explicit formula of definition for the inverse function $\operatorname{arctanh}: (-1, 1) \to \mathbb{R}$ of the function tanh.

1f) Find the first derivative of arctanh.

Proof 1c). Let $y \in (-1, 1)$. Suppose that $y = \tanh x$, then we have

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$(e^x + e^{-x}) \cdot y = e^x - e^{-x}$$

$$(1+y) e^{-x} = (1-y) e^x$$

$$e^{2x} = \frac{1+y}{1-y} \quad (-1 < \tanh < 1,\ y \neq 1)$$

$$x = \frac{1}{2} \log \frac{1+y}{1-y}.$$

Hence, a solution of the equation $y = \tanh x$ with unknown x is given by $x = \frac{1}{2} \log \frac{1+y}{1-y}$.

1e) By the statement of 1c), the explicit formula of $\operatorname{arctanh} x$ is.

$\operatorname{arctanh} x = \frac{1}{2} \log \frac{1+x}{1-x}$ for any $x \in (-1, 1)$.

1f). For any $x \in (-1, 1)$, we have that

$$\frac{d}{dx}(\operatorname{arctanh} x) = \frac{d}{dx}\left[\frac{1}{2} \log \frac{1+x}{1-x}\right]$$

$$= \frac{1}{2} \frac{d}{dx}\left[\log(1+x) - \log(1-x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{1+x} - \frac{-1}{1-x}\right] = \frac{1}{1-x^2}.$$

Hence, the first derivative of arctanh is $\frac{1}{1-x^2}$ for all $x \in (-1, 1)$. $\square$