Title ____________________

Math1010F

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Ex1.1# Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \cos x, & \text{if } x \geq 0; \\ 1, & \text{if } x < 0. \end{cases}$$

Is $f(x)$ differentiable at $x=0$?

Proof: If $f(x)$ differentiable at $x=0$, $f'_+(x)$ and $f'_-(x)$ exist and equal.

Noting that

$$\lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\cos(0+h)-\cos 0}{h} = \lim_{h\to 0^+} \frac{\cos h - 1}{h} = 0$$

Not

and

$$\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{\cos(0+h)-\cos 0}{h} = \lim_{h\to 0^-} \frac{\cos h - 1}{h} = 0.$$

so we have that $f(x)$ differentinble at $x=0$, and $f'(0)=0$. □

Note. How to compute the limit $\lim_{h\to 0} \frac{\cos h - 1}{h}$?

Method I: By the definition of $\cos x$.

Since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, it holds that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{1}{h}\left[-1+1-\frac{h^2}{2!}+\frac{h^4}{4!}+\cdots\right]$$

$$= \lim_{h\to 0}\left[-\frac{h}{2!}+\frac{h^3}{4!}+\cdots\right] = 0. \quad □$$

Method II: By the formula $1 = \sin^2 t + \cos^2 t$:

Since $(1-\cosh)(1+\cos h) = \sin^2 h$, it holds that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{(\cos h-1)(\cos h+1)}{h(\cos h+1)} = \lim_{h\to 0} \frac{\sin^2 h}{h(\cos h+1)}$$

$$= \lim_{h\to 0} \frac{h}{\cos h+1} \cdot \lim_{h\to 0}\left(\frac{\sin h}{h}\right)^2 = 0. \qquad \square$$

Method III: By the formula $\cos(2x) = 1 - 2\sin^2 x$:

Since $\cos h = 1 - 2\sin^2\left(\frac{h}{2}\right)$, it holds that

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{1-2\sin^2\left(\frac{h}{2}\right)-1}{h} = \lim_{h\to 0} \frac{h\cdot \sin^2\left(\frac{h}{2}\right)}{-2\left(\frac{h}{2}\right)^2}$$

$$= \lim_{h\to 0}\left(-\frac{h}{2}\right) \cdot \lim_{h\to 0}\left[\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right]^2 = 0. \qquad \square$$

Ex1.2#. Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^3, & \text{if } x \leq 1; \\ ax+b & \text{if } x > 1. \end{cases}$$

If $f(x)$ differentiable at $x=1$, find the values of a and b.

Analysis: Since two variables need to determined, we need to obtain two conditions from differentiability. If $f(x)$ differentiable at $x=1$, it holds that

a. $f(x)$ is continuous at $x=1$; (Theorem)

b. $f'_+(1)$ and $f'_-(1)$ exist and are equal. (Equivalent definition)

Proof. If $f(x)$ differentiable at $x=1$, $\overset{a.}{}$ $f(x)$ is continuous at $x=1$. Noting that

$$\begin{cases} \lim_{x\to1^+} f(x) = \lim_{x\to1^+} (ax+b) = a+b; \\ \lim_{x\to1^-} f(x) = \lim_{x\to1^-} x^3 = 1. \end{cases}$$

Hence, by $\lim_{x\to1^+} f(x) = \lim_{x\to1^-} f(x)$, it holds that $a+b=1$. $(*)$

Besides, $\overset{b.}{}$ $f'_+(1)$ and $f'_-(1)$ exist and are equal. Noting that

$$\begin{cases} \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^+} \frac{ax+b-1}{x-1} = \lim_{x\to1^+} \frac{-(b-1)x+b-1}{x-1} = \lim_{x\to1^+} (1-b) = 1-b; \text{ (by } (*)) \\ \lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x\to1^-} \frac{x^3-1}{x-1} = \lim_{x\to1} (x^2+x+1) = 3. \end{cases}$$

that is, $1-b=3$. Therefore, we have $a=3$ and $b=-2$. $\square$

Ex13# Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \frac{1-\cos x}{x^2}, & \text{if } x<0; \\ ax+b, & \text{if } x \geq 0. \end{cases}$$

If $f(x)$ differentiable at $x=0$, find the values of a and b.

Proof. If $f(x)$ differentiable at $x=0$, a. $f(x)$ is continuous at $x=0$. Noting that

$$\begin{cases} \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (ax+b) = b; \\ \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1-\cos x}{x^2} = \frac{1}{2}. \end{cases}$$

Hence, by $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$, it holds that $b = \frac{1}{2}$.

Besides, b. $f'_+(0)$ and $f'_-(0)$ exist and are equal. Noting that

$$\begin{cases} \lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{(ax+b)-b}{x} = a; \\ \lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^-} \frac{x^{-2}(1-\cos x)-b}{x} = \underline{\lim_{x\to 0^-} \frac{2(1-\cos x)-x^2}{2x^3}} = 0; \end{cases}$$

that is, $a=0$. Therefore, we have $a=0$ and $b=\frac{1}{2}$. □

Note. How to compute the limit $\lim_{x\to 0} \frac{2(1-\cos x)-x^2}{2x^3}$?

Proof. Since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ (by definition), it holds that

$$\lim_{x\to 0} \frac{2(1-\cos x)-x^2}{2x^3} = \lim_{x\to 0} \frac{2\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} \cdots\right)-x^2}{2x^3} = \lim_{x\to 0} \frac{2\left(-\frac{x^4}{4!} + \frac{x^6}{6!} - \cdots\right)}{2x^3} = 0,$$

that is, $\lim_{x\to 0} \frac{2(1-\cos x)-x^2}{2x^3} = 0$. □

Ex1.4# Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

(a) Use the definition of the derivative to find $f'(0)$.

(b) Use the definition of the derivative to find $f'(x)$ for x not equal 0.

Proof: (a) Using the definition of the derivative, we find that

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(h)-f(0)}{h} \\ &= \lim_{x\to 0} \frac{h^2 \sin(\frac{1}{h}) - 0}{h} \\ &= \lim_{x\to 0} h \cdot \sin\frac{1}{h} \qquad \left(-h \leq h \cdot \sin\frac{1}{h} \leq h\right) \\ &= 0, \end{aligned}$$

by the Sandwich Theorem. $\square$

(b) Since $f(x) = x^2 \sin\frac{1}{x}$ $(x \neq 0)$, we have that

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[(x+h)^2 \sin\frac{1}{x+h} - x^2 \sin\frac{1}{x} \right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\overset{a.}{\underline{x^2\left(\sin\frac{1}{x+h} - \sin\frac{1}{x}\right)}} + \overset{b.}{\underline{2hx \cdot \sin\frac{1}{x+h}}} + \overset{c.}{\underline{h^2 \sin\frac{1}{x+h}}} \right]. \end{aligned}$$

Noting that $\lim_{h\to 0} 2x \cdot \sin\frac{1}{x+h} = 2x\sin\frac{1}{x}$ ${}^{b.}$, $\lim_{h\to 0} h \cdot \sin\frac{1}{x+h} = 0$ ${}^{c.}$ and

$$\begin{aligned} \lim_{h\to 0} \frac{x^2}{h}\left(\sin\frac{1}{x+h} - \sin\frac{1}{x}\right)^{a.} &= \lim_{h\to 0} \frac{2x^2}{h} \cos\frac{2x+h}{2x(x+h)} \sin\frac{-h}{2x(x+h)} \\ &= \lim_{h\to 0} \left[\cos\frac{2x+h}{2x(x+h)}\right] \cdot \left[\frac{2x^2}{-2x(x+h)}\right] \cdot \left[\frac{\sin\frac{-h}{2x(x+h)}}{\frac{-h}{2x(x+h)}}\right] \\ &= \left[\lim_{h\to 0} \cos\frac{2x+h}{2x(x+h)}\right] \cdot \left[\lim_{h\to 0} \frac{2x^2}{-2x(x+h)}\right] \cdot \left[\lim_{h\to 0} \frac{\sin\frac{-h}{2x(x+h)}}{\frac{-h}{2x(x+h)}}\right] \\ &= \cos\frac{1}{x} \cdot (-1) \times 1 = -\cos\frac{1}{x}. \end{aligned}$$

so $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ for $x \neq 0$. $\square$

Ex1.4# Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

1(a) Use the definition of the derivative to find $f'(0)$.

1(b) Use the definition of the derivative to find $f'(x)$ for x not equal 0.

1(c) Determine the explicit formula for the function $f'(x)$. Is $f'(x)$ continuous at 0?

Proof: 1(c) By the 1(a) and 1(b), we have that

$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

Noting that $\lim_{x\to 0} \cos\frac{1}{x}$ does not exist (Why?), then $\lim_{x\to 0} f'(x)$ does not exist.

Hence, $f'(x)$ is not continuous at 0. $\square$

Note. Why $\lim_{x\to 0} \cos\frac{1}{x}$ does not exist? (Draw picture).

Theorem. (Sequential criterion for limits of functions)

Let $f(x)$ be a real valued function. Then

$$\lim_{x\to a} f(x) = L$$

if and only if for any sequence x_n of real numbers with $x_n \neq a$ for any n and $\lim_{n\to+\infty} x_n = a$, we have

$$\lim_{n\to+\infty} f(x_n) = L.$$

Proof. Let $f(x) = \cos\frac{1}{x}$. Suppose that $\lim_{x\to 0} f(x)$ exist.

Let $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{2n\pi+\pi}$. Since $\lim_{n\to+\infty} x_n = 0$ and $\lim_{n\to+\infty} y_n = 0$, by the sequential criterion for limits of functions, it holds that

$$\begin{cases} \lim_{n\to+\infty} f(x_n) = \lim_{x\to 0} f(x); \\ \lim_{n\to+\infty} f(y_n) = \lim_{x\to 0} f(x). \end{cases}$$

that is, $\lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} f(y_n)$.

On the other hand, $f(x_n) = \cos\frac{1}{x_n} = \cos(2n\pi) = 1$, for all $n \in \mathbb{Z}_+$ and $f(y_n) = \cos\frac{1}{y_n} = \cos(2n\pi+\pi) = -1$, for all $n \in \mathbb{Z}_+$. Hence $\{f(x_n)\}$ and $\{f(y_n)\}$ are both the constant sequence, and it holds that

$$\begin{cases} \lim_{n\to+\infty} f(x_n) = 1; \\ \lim_{n\to+\infty} f(y_n) = -1, \end{cases}$$

that is, $\lim_{n\to+\infty} f(x_n) \neq \lim_{n\to+\infty} f(y_n)$, which is the contradiction. Hence, $\lim_{x\to 0} \cos\frac{1}{x}$ does not exist. $\square$

Method II: By the definition of limit.

Proof. If $\lim_{x\to 0} \cos\frac{1}{x}$ exist, say $\lim_{x\to 0} \cos\frac{1}{x} = L$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - 0| < \delta$ implies $|\cos\frac{1}{x} - L| < \varepsilon$. Let $\varepsilon = \frac{1}{2}$, $y = \frac{1}{2n\pi}$ and $z = \frac{1}{2n\pi+\pi}$. We can find large enough n such that $0 < |y| < \delta$ and $0 < |z| < \delta$, so it holds that

$$\begin{cases} |\cos\frac{1}{y} - L| < \varepsilon; \\ |\cos\frac{1}{z} - L| < \varepsilon. \end{cases} \quad \text{that is} \quad \begin{cases} \frac{1}{2} < L < \frac{3}{2}; \\ -\frac{3}{2} < L < -\frac{1}{2}. \end{cases}$$

which is a contradiction. Hence, $\lim_{x\to 0} \cos\frac{1}{x}$ does not exist. $\square$

Ex 15# Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0; \\ 0, & x = 0. \end{cases}$$

(a) Is $f(x)$ continuous at 0? Justify your answer.

(b) Is $f(x)$ differentiable at 0? Justify your answer.

Proof. (a) By Sandwich Theorem, we have that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

that is, $\lim_{x\to 0} f(x) = f(0)$. Hence, $f(x)$ is continuous at $x=0$. $\square$

(b) If $f(x)$ is differentiable at 0, $f'_+(0)$ and $f'_-(0)$ exist and are equal. However, noticing that

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \sin\frac{1}{x}$$ (See Page 8)

does not exist. Hence, f is not differentiable at 0. $\square$

Ex 1.6# Let a be a real number and $f(x)$ be a function defined by

$$f(x) = \lim_{n\to\infty} \frac{a(n^x - n^{-x})}{n^x + n^{-x}}.$$

(a) Find $f(0)$.

(b) Show that $f(x)$ is a constant for $x>0$ and $f(x)$ is another constant for $x<0$.

Proof: (a) Let $x=0$, we have that

$$f(0) = \lim_{n\to\infty} \frac{a(n^0 - n^0)}{n^0 + n^0} = 0,$$

that is, $f(0) = 0$. □

(b) If $x>0$, it holds that

$$\begin{aligned} f(x) &= \lim_{n\to+\infty} \frac{a(n^x - n^{-x})}{n^x + n^{-x}} \\ &= \lim_{n\to+\infty} \frac{a(1-n^{-2x})}{1+n^{-2x}} \\ &= a. \end{aligned}$$

If $x<0$, it holds that

$$\begin{aligned} f(x) &= \lim_{n\to+\infty} \frac{a(n^x - n^{-x})}{n^x + n^{-x}} \\ &= \lim_{n\to+\infty} \frac{a(n^{2x}-1)}{n^{2x}+1} \\ &= -a. \end{aligned}$$

Hence, $f(x)$ is a constant for $x>0$ and $f(x)$ is another constant for $x<0$. □

Ex 1.6# Let a be a real number and $f(x)$ be a function defined by

$$f(x) = \lim_{n\to\infty} \frac{a(n^x - n^{-x})}{n^x + n^{-x}}.$$

(a) Find $f(0)$.

(b) Show that $f(x)$ is a constant for $x>0$ and $f(x)$ is another constant for $x<0$.

(c) If $f(x)$ is continuous at $x=0$, find the value(s) of a.

Proof: (c) By (a) and (b). we can rewrite $f(x)$ as

$$f(x) = \begin{cases} a, & \text{if } x>0; \\ 0, & \text{if } x=0; \\ -a, & \text{if } x<0. \end{cases}$$

If $f(x)$ is continuous at $x=0$, it holds that

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x),$$

that is, $a=-a$. Hence, we find that $a=0$. $\square$

Ex 1.17# Here you may take for granted that

$$\lim_{t\to 0}\frac{\sin(\alpha t)}{\alpha t}=1,$$

for every non-zero real number α.

(a) Use definition to find the first derivative of $\sin x$.

(b) Use definition to find the first derivative of $\cos x$.

Proof: (a) By the first principles, it holds that

$$\begin{aligned}\lim_{t\to 0}\frac{\sin(x+t)-\sin x}{t} &= \lim_{t\to 0}\frac{1}{t}\left[\sin x\cos t+\cos x\sin t-\sin x\right]\\ &= \lim_{t\to 0}\frac{1}{t}\left[\sin x(\cos t-1)+\cos x\sin t\right]\\ &= \lim_{t\to 0}\left[\sin x\cdot\frac{\cos t-1}{t}+\cos x\cdot\frac{\sin t}{t}\right]\\ &= \sin x\cdot\underbrace{\lim_{t\to 0}\frac{\cos t-1}{t}}_{=0}+\cos x\cdot\lim_{t\to 0}\frac{\sin t}{t}\\ &= \cos x.\end{aligned}$$

In Ex 1.1# Note.

Hence, the first derivative of $\sin x$ is $\cos x$. $\square$

(b) By the first principles, it holds that

$$\begin{aligned}\lim_{t\to 0}\frac{\cos(x+t)-\cos x}{t} &= \lim_{t\to 0}\frac{1}{t}\left[\cos x\cos t-\sin x\sin t-\cos x\right]\\ &= \lim_{t\to 0}\frac{1}{t}\left[\cos x(\cos t-1)-\sin x\cdot\sin t\right]\\ &= \lim_{t\to 0}\left[\cos x\cdot\frac{\cos t-1}{t}-\sin x\cdot\frac{\sin t}{t}\right]\\ &= \cos x\cdot\lim_{t\to 0}\frac{\cos t-1}{t}-\sin x\cdot\lim_{t\to 0}\frac{\sin t}{t}\\ &= -\sin x.\end{aligned}$$

Hence, the first derivative of $\cos x$ is $-\sin x$. $\square$

Ex1.8# Let A be constant, and $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = e^{|x|} - A|x|,$$

for any $x \in \mathbb{R}$.

(a) Show that f is continuous at 0.

Proof: (a) Noting that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(e^{|x|} - A|x|\right) = \lim_{x\to 0} e^{|x|} - A \cdot \lim_{x\to 0} |x| = e^0 = 1,$$

and $f(0) = 1$, that is $\lim_{x\to 0} f(x) = f(0)$. Hence, f is continuous at 0. $\square$

Method II: Composite function.

Proof: Let $g: \mathbb{R} \to \mathbb{R}$ be the function defined by $h(x) = e^x - Ax$ for any $x \in \mathbb{R}$. Then, we obtain that $g \circ \text{abs}(x) = g(|x|) = f(x)$, for any $x \in \mathbb{R}$. Since abs is continuous at 0 and g is continuous at $\text{abs}(0) = 0$, f is continuous at 0. $\square$

Ex18# Let A be constant, and $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = e^{|x|} - A|x|.$$

for any $x \in \mathbb{R}$.

(a) Show that f is continuous at 0.

(b) Suppose that f is differentiable at 0.

A. Find the value of A and the value of $f'(0)$.

Proof: (b) A. Since $f(x) = e^{|x|} - A|x|$, $f(x)$ can be rewritten as

$$f(x) = \begin{cases} e^{-x} + Ax, & \text{if } x < 0; \\ 1, & \text{if } x = 0; \\ e^{x} - Ax, & \text{if } x > 0. \end{cases}$$

Since $f(x)$ is differentiable at 0, $f'_+(0)$ and $f'_-(0)$ exist and are equal.

Noting that

$$\begin{cases} f'_+(0) = \lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x \to 0^+} \dfrac{e^x - Ax - 1}{x} = -A + \lim\limits_{x \to 0^+} \dfrac{e^x - 1}{x} = -A + 1; \\ f'_-(0) = \lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x} = \lim\limits_{x \to 0^-} \dfrac{e^{-x} + Ax - 1}{x} = A - \lim\limits_{x \to 0^-} e^{-x} \cdot \dfrac{e^x - 1}{x} = A - 1, \end{cases}$$

So we have that $-A + 1 = A - 1$, that is, $A = 1$. And then, it holds that $f'_+(0) = 0$ and $f'_-(0) = 0$, that is, $f'(0) = 0$. □

Note. How to compute the limit $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x}$?

Proof. Noting that $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ (by definition), so $e^x - 1 = x[1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots]$, that is $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = \lim\limits_{x \to 0} [1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots] = 1$. □

Ex 18[#] Let A be constant, and $f: \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = e^{|x|} - A|x|,$$

for any $x \in \mathbb{R}$.

(a) Show that f is continuous at 0.

(b) Suppose that f is differentiable at 0.

A. Find the value of A and the value of $f'(0)$.

B. Is f twice differentiable at 0? Justify your answer. Is yes, also find the value of $f''(0)$.

Proof: (b) B. By the statement in (a) and (b) A. we have that

$$f(x) = \begin{cases} e^{-x} + x, & \text{if } x < 0; \\ 1, & \text{if } x = 0; \\ e^{x} - x, & \text{if } x > 0, \end{cases} \quad \text{and} \quad f'(x) = \begin{cases} -e^{-x} + 1, & \text{if } x < 0; \\ 0, & \text{if } x = 0; \\ e^{x} - 1, & \text{if } x > 0. \end{cases}$$

If f is twice differentiable at 0, $f''_+(0)$ and $f''_-(0)$ exist and are equal.

Noting that

$$\begin{cases} f''_+(0) = \lim\limits_{x \to 0^+} \dfrac{f'(x) - f'(0)}{x} = \lim\limits_{x \to 0^+} \dfrac{e^x - 1}{x} = 1; \\ f''_-(0) = \lim\limits_{x \to 0^-} \dfrac{f'(x) - f'(0)}{x} = \lim\limits_{x \to 0^-} \dfrac{-e^{-x} + 1}{x} = \lim\limits_{x \to 0^-} e^{-x} \cdot \dfrac{e^x - 1}{x} = 1. \end{cases}$$

that is $f''_+(0) = f''_-(0)$. Hence, f is twice differentiable at 0. Besides, we have $f''_+(0) = f''_-(0) = 1$, so $f''(0) = 1$. $\square$

PreCW3 Ex2(c) Compute the limit as following:

$$\lim_{x\to 0} \frac{1-\cos x\cos(2x)\cos(3x)}{1-\cos x}.$$

Analysis. For complete (composite function) limits, we hope to simplify the limits by tranforming the variables. At this problem show, we hope to transform the limit into $\lim_{x\to 0} f(\cos x)$.

Proof. Noting that $\cos(2x) = 2\cos^2 x - 1$ and

$$\begin{aligned}\cos(3x) &= \cos(2x+x)\\ &= \cos(2x)\cos x - \sin(2x)\sin x\\ &= (2\cos^2 x-1)\cos x - 2\sin x\cos x\sin x\\ &= (2\cos^2 x-1)\cos x - 2\cos x(1-\cos^2 x)\\ &= 4\cos^3 x - 3\cos x.\end{aligned}$$

Let $t=\cos x$. Since $t\to 1$ as $x\to 0$, it holds that

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x\cos(2x)\cos(3x)}{1-\cos x} &= \lim_{x\to 0}\frac{1-\cos x[2\cos^2 x-1][4\cos^3 x-3\cos x]}{1-\cos x}\\ &= \lim_{t\to 1}\frac{1-t(2t^2-1)(4t^3-3t)}{1-t} \quad (s=1-t)\\ &= \lim_{s\to 0}\frac{1}{s}[1-(1-s)(2s^2-4s+1)(-4s^3+12s^2-9s+1)]\\ &= \lim_{s\to 0}\frac{1}{s}[1-(1-14s+\_s^2+\_s^3+\cdots)]\\ &= 14.\end{aligned}$$

Constant term: $1\times1\times1=1$

linear term ($\_s$):

$-s\times1\times1-4s\times1\times1-9s\times1\times1=-14s.$

$\square$

PreCW3 Ex2 (d) Compute the limit as following:

$$\lim_{x\to 0} \frac{\sqrt{1+\sin x}-1}{e^x-1}.$$

Proof: We have that

$(1+\sin x)-1=\sin x$

$$\lim_{x\to 0} \frac{\sqrt{1+\sin x}-1}{e^x-1} = \lim_{x\to 0} \frac{(\sqrt{1+\sin x}-1)(\sqrt{1+\sin x}+1)}{(e^x-1)(\sqrt{1+\sin x}+1)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{1+\sin x}+1} \cdot \lim_{x\to 0} \frac{\sin x}{e^x-1}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{x}{e^x-1}$$

$$= \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{x}{e^x-1}$$

$$= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}.$$

□