Title

Ex1.1# For what value of the constant c is the function f continuous on the interal $(-\infty, +\infty)$?

$$f(x) = \begin{cases} x^2 - 10, & x \leq c; \\ 8x - 26, & x > c. \end{cases}$$

Proof: Suppose that f be continous at c. Then.

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x).$$

By solving the equation

$$c^2 - 10 = 8c - 26.$$

Hence, we obtain $c = 4$. $\square$

Ex12# Let $f(x) = |x-7|$. Evaluate the following limits,

$$\lim_{x\to 7^-} \frac{f(x)-f(7)}{x-7}, \quad \text{and} \quad \lim_{x\to 7^+} \frac{f(x)-f(7)}{x-7}.$$

Thus the function $f(x)$ is not differentiable at 7.

Prof: Let $f(x) = |x-7|$. We have that

$$\lim_{x\to 7^-} \frac{f(x)-f(7)}{x-7} = \lim_{x\to 7^-} \frac{(7-x)-0}{x-7} = -1;$$

and

$$\lim_{x\to 7^+} \frac{f(x)-f(7)}{x-7} = \lim_{x\to 7^+} \frac{(7-x)-0}{x-7} = 1.$$

Thus, $f(x)$ is not differentiable at 7. $\square$

Ex1.3# Let $f(x)=\begin{cases} -4x^2+4x, & x<0; \\ 3x^2-5, & x\geq 0. \end{cases}$

Using the definition of the derivative, compute $f'(0)$.

Proof: Since

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{-4x^2+4x+5}{x}=\lim_{x\to 0^-}\left(4+\frac{5}{x}\right)=-\infty$$

does not exist. Hence, f is not differentiable at $x=0$. □

Method II: Differentiable ⇒ Continuous

Proof: Since $\lim_{x\to 0^-}f(x)=0$ and $\lim_{x\to 0^+}f(x)=-5$ are different, $f(x)$ is no continuous at $x=0$.

Suppose that $f(x)$ is differentiable at $x=0$. It is obvious that f is continuous at $x=0$, which is a contradiction. Hence, f is no differentiable at $x=0$. □

Ex14# Find the value of the constant a that makes the following function continuous on $(-\infty, +\infty)$?

$$f(x) = \begin{cases} \dfrac{6x^3+5x^2+7x+8}{x+1}, & x < -1; \\ 5x^2 - x + a, & x \geq -1. \end{cases}$$

Proof: It is obvious that $6x^3+5x^2+7x+8 = (x+1)(6x^2-x+8)$. that is

$$f(x) = \begin{cases} 6x^2 - x + 8, & x < -1; \\ 5x^2 - x + a & x \geq -1. \end{cases}$$

It suffices to show that $f(x)$ is continuous at $x=-1$. that is,

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$$

Noting that

$$\begin{cases} \lim_{x\to -1^-} f(x) = 6+1+8 = 15; \\ \lim_{x\to -1^+} f(x) = 5-1+a = a+4, \end{cases}$$

we have that $a = 9$. $\square$

Ex1.5# Let $f(x)=e^{2x}$. Use the limit definition of the derivative to find $f'(x)$.

Proof: By the definition, we have that

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{2(x+h)}-e^{2x}}{h}=2e^{2x}\lim_{h\to 0}\frac{e^{2h}-1}{2h}=2e^{2x}.$$

Hence, $f'(x)=2e^{2x}$. $\square$

Ex1.6# Use the definition of a derivative to find $f'(x)$ and $f'(0)$ where

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & x \neq 0; \\ 0, & x = 0. \end{cases}$$

Proof: Using the definition of the derivative we find that

$$f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h-0} = \lim_{h\to 0} h\sin\frac{1}{h} = 0.$$

where we use Sandwich Theorem. $(0 \leq h\sin\frac{1}{h} \leq h)$

For $x \neq 0$, we have that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 \sin\frac{1}{x+h} - x^2\sin\frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\Big[\underbrace{x^2\Big(\sin\frac{1}{x+h} - \sin\frac{1}{x}\Big)}_{\mathrm{I}} + \underbrace{2hx\sin\frac{1}{x+h}}_{\mathrm{II}} + \underbrace{h^2\sin\frac{1}{x+h}}_{\mathrm{III}} \Big].$$

Noting that $\lim_{h\to 0} 2x\sin\frac{1}{x+h} = 2x\sin\frac{1}{x}$ (II), $\lim_{h\to 0} h\sin\frac{1}{x+h} = 0$ (III) and

$$\lim_{h\to 0} \frac{x^2}{h}\Big(\sin\frac{1}{x+h} - \sin\frac{1}{x}\Big) = \lim_{h\to 0} \frac{2x^2}{h} \cos\frac{2x+h}{2x(x+h)} \sin\frac{-h}{2x(x+h)}$$

$$= \lim_{h\to 0} \cos\frac{2x+h}{2x(x+h)} \lim_{h\to 0} \frac{2x^2}{h} \sin\frac{-h}{2x(x+h)}$$

$$= \cos\frac{1}{x} \lim_{h\to 0}\left[\frac{2x^2}{-2x(x+h)} \frac{\sin\left(\frac{-h}{2x(x+h)}\right)}{\frac{-h}{2x(x+h)}} \right]$$

$$= \cos\frac{1}{x} \cdot (-1) \times 1 = \cos\frac{1}{x} \quad (\mathrm{I}),$$

Hence, $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x} \quad (x \neq 0)$ $\square$

Ex1.7# Determine, for each of the limits below, whether it exists or not. Give an appropriate justification. Where the limits exists, also evaluate its value.

$$\lim_{x\to+\infty} x\left(e^{\frac{1}{x}}-1\right).$$

e^t

Proof: Let $t=x^{-1}$. We have that

$$\lim_{x\to+\infty} x\left(e^{\frac{1}{x}}-1\right) = \lim_{t\to0^+}\frac{e^t-1}{t}=1.$$

$\square$

Ex1.7# Determine, for each of the limits below, whether it exists or not. Give an appropriate justification. Where the limits exists, also evaluate its value.

$$\lim_{x\to+\infty} x\log\frac{x+1}{x}.$$

Proof. We have that

$$\lim_{x\to+\infty} x\log\frac{x+1}{x} = \lim_{x\to+\infty}\log\left(1+\frac{1}{x}\right)^x$$

$$= \log\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$$

$g(u)=\log u$ is continuous

$$= \log e = 1.$$

□

Ex1.7# Determine, for each of the limits below, whether it exists or not. Give an appropriate justification. Where the limits exists, also evaluate its value.

$$\lim_{x\to+\infty}(x^2+x+1)\sin\frac{1}{x^2}.$$

Proof: Let $x=t^{-1}$. We have that

$$\lim_{x\to+\infty}(x^2+x+1)\sin\frac{1}{x^2} = \lim_{t\to0^+}\frac{1+t+t^2}{t^2}\sin t^2$$

If $\lim f(x)$ and $\lim g(x)$ exist, $\lim f(x)g(x)=\lim f(x)\cdot\lim g(x)$

$$= \lim_{t\to0^+}(1+t+t^2)\frac{\sin t^2}{t^2}$$

$$= \lim_{t\to0^+}(1+t+t^2)\lim_{t\to0^+}\frac{\sin t^2}{t^2}$$

$$= 1\times1=1,$$

where we use that $\lim_{u\to0}\frac{\sin u}{u}=1$. □

Ex 1.8# 1(a) Find $\lim_{x\to 0}\left(\frac{a^x+b^x+1}{3}\right)^{\frac{1}{x}}$

where $a, b>0$.

Analyse: Since $\lim_{x\to 0}(a^x+b^x+1)=3$ and $\lim_{x\to 0}\frac{1}{x}=\infty$, the limit is of type 1^∞.
Hence, we can try to use $\lim_{x\to +\infty}(1+\frac{1}{x})^x=e$ to compute it.

Proof: We have that

$$I=\lim_{x\to 0}\left(\frac{a^x+b^x+1}{3}\right)^{\frac{1}{x}}=\lim_{x\to 0}\left(1+\frac{a^x+b^x-2}{3}\right)^{\frac{1}{x}}.$$

Let $f(x)=\frac{1}{3}(a^x+b^x-2)$. It is obvious that $f(x)\to 0$ as $x\to 0$.
Next, we will create $[1+f(x)]^{\frac{1}{f(x)}}$.

Noting that

$$I=\lim_{x\to 0}[1+f(x)]^{\frac{1}{f(x)}\cdot\frac{f(x)}{x}}=\lim_{x\to 0}[1+f(x)]^{\frac{1}{f(x)}\cdot\lim_{x\to 0}\frac{f(x)}{x}} \quad (*)$$

where we use that $\lim f(x)^{g(x)}=\lim f(x)^{\lim g(x)}$. (Check!)
Since $\lim_{x\to 0}[1+f(x)]^{\frac{1}{f(x)}}=e$ (since $\lim_{x\to 0}f(x)=0$), it suffies to consider the limit $\lim_{x\to 0}\frac{f(x)}{x}$.

By the definition of $f(x)$, we have that

$$\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}\frac{a^x+b^x-2}{3x}=\lim_{x\to 0}\left[\frac{a^x-1}{3x}+\frac{b^x-1}{3x}\right].$$

Here, we let $\lim_{x\to 0}(a^x-1)=0$ and $\lim_{x\to 0}(b^x-1)=0$.

Since $a^x = e^{x\log a}$

$$\lim_{x\to 0}\frac{a^x-1}{3x} = \frac{\log a}{3}\lim_{x\to 0}\frac{e^{x\log a}-1}{x\log a} = \frac{\log a}{3}\times 1 = \frac{\log a}{3},$$

Similarly, we have that $\lim_{x\to 0}\frac{b^x-1}{3x} = \frac{1}{3}\log b$, that is

$$\lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{a^x-1}{3x} + \lim_{x\to 0}\frac{b^x-1}{3x} = \frac{\log a+\log b}{3} = \log(ab)^{\frac{1}{3}}.$$

Hence, by (*), it holds that

$$I = \lim_{x\to 0}\left(\frac{a^x+b^x+1}{3}\right)^{\frac{1}{x}} = \exp\left[\log(ab)^{\frac{1}{3}}\right] = \sqrt[3]{ab}.$$

$\square$

Method II: L' Hospital Law (Study in the furture)

Proof. We have that

$$\lim_{x \to 0}\left(\frac{a^x+b^y+1}{3}\right)^{\frac{1}{x}} = \exp\left[\lim_{x\to 0} \frac{\log\left(\frac{a^x+b^x+1}{3}\right)}{x}\right]$$

$$= \exp\left[\lim_{x\to 0} \frac{a^x \log a + b^x \log b}{a^x+b^x+1}\right]$$ L'Hospital Law

$$= \exp\left[\frac{\log a + \log b}{3}\right] = \sqrt[3]{ab} \qquad \square$$

Ex1.8 1(b) Let $c_1, c_2, \cdots, c_n$ be n positive numbers. Find

$$\lim_{x\to 0}\left(\frac{c_1^x+c_2^x+\cdots+c_n^x}{n}\right)^{\frac{1}{x}}.$$

Proof: We have that

$$\lim_{x\to 0}\left(\frac{c_1^x+c_2^x+\cdots+c_n^x}{n}\right)^{\frac{1}{x}} = \exp\left[\lim_{x\to 0}\frac{\log\left(\frac{c_1^x+c_2^x+\cdots+c_n^x}{n}\right)}{x}\right]$$

$$= \exp\left[\lim_{x\to 0}\frac{c_1^x\log c_1+c_2^x\log c_2+\cdots\cdots+c_n^x\log c_n}{c_1^x+c_2^x+\cdots+c_n^x}\right]$$

$$= \exp\left[\frac{1}{n}\log(c_1c_2\cdots c_n)\right] = \sqrt[n]{c_1c_2\cdots c_n}. \qquad \square$$

The proof can also be completed using Method I.

Ex 1.9* Let $f(x)$ be a continuous function defined for $x>0$ and for any $x, y>0$

$$f(xy) = f(x) + f(y).$$

(a) Find $f(1)$

Proof. Let $x=1$ and $y=1$. It holds that

$$f(1) = f(1) + f(1),$$

that is, $f(1) = 0$. $\square$

Ex1.9* Let $f(x)$ be a continuous function defined for $x>0$ and for any $x, y>0$

$$f(xy)=f(x)+f(y)$$

(a) Find $f(1)$

(b) Let a be a positive real number. Prove that for any rational number r, $f(a^r)=rf(a)$.

Analyse: rational number $r \Leftrightarrow \exists\, m, n \in \mathbb{Z}$ and $n \neq 0$ s.t. $r=\frac{m}{n}$

I. $r>0$ positive: $\begin{cases} n=1, \text{ we prove that } f(a^m)=mf(a). \\ m=1, \text{ we prove that } f(a^{\frac{1}{n}})=\frac{1}{n}f(a). \end{cases}$

II. $r<0$ negative.

III. $r=0$.

Prof: I Consider that $r>0$.

① If $n=1$, we will prove that $f(a^m)=mf(a)$. We have

$$\begin{aligned} f(a^m) &= f(a^{m-1}\cdot a) \\ &= f(a^{m-1})\cdot f(a) \quad \text{(By definition)} \\ &= f(a^{m-2})\cdot f(a)\cdot f(a) \\ &= \cdots\cdots \\ &= f(a)\cdot f(a)\cdots f(a) = [f(a)]^m. \end{aligned}$$

② If $m=1$, we will prove that $f(a^{\frac{1}{n}})=\frac{1}{n}f(a)$. We have

$$f(a) = f[(a^{\frac{1}{n}})^n] = n\cdot f(a^{\frac{1}{n}}), \quad \text{(By ①)}$$

that is, $f(a^{\frac{1}{n}})=\frac{1}{n}f(a)$

③ Now, let us consider $r = \frac{m}{n}$. It holds that

$$f(a^{\frac{m}{n}}) = f[(a^{\frac{1}{n}})^m]$$
$$= m \cdot f(a^{\frac{1}{n}}) \quad \text{(By ①)}$$
$$= m \cdot \frac{1}{n} f(a), \quad \text{(By ②)}$$

that is, $f(a^r) = r f(a)$, where $r = \frac{m}{n}$.

II Consider that $r<0$. By Step I, we know that $f(a^p) = p f(a)$, if p is a positive rational number. Since $-r$ is a positive rational number, we have that $f(a^{-r}) = -r f(a)$. (*)

On the other hand, by the definition of $f(x)$,

$$f(a^{-r}) + f(a^r) = f(a^r \cdot a^{-r}) = f(1) = 0,$$

this, combining with (*), implies that

$$f(a^r) = -f(a^{-r}) = -[-r f(a)] = r f(a).$$

III Consider that $r=0$. Since $a^r = a^0 = 1$, we have that

$$f(a^0) = f(1) = 0 = 0 \cdot f(a)$$

where we use the rusult of (a). ($f(1)=0$)

Therefore, combining Step I, II and III, we complete the proof. □

Ex 1.9* Let $f(x)$ be a continuous function defined for $x>0$ and for any $x, y>0$

$$f(xy) = f(x) + f(y).$$

(a) Find $f(1)$

(b) Let a be a positive real number. Prove that for any rational number r, $f(a^r) = rf(a)$.

(c) Show that for all $x>0$, $f(a^x) = xf(a)$, where a is a positive real constant. Hence, prove that for all $x>0$, $f(x) = c\log x$, where c is a constant.

Proof: First, we prove that for all $x>0$, $f(a^x) = xf(a)$.

For all $x \in \mathbb{R}$, there exists a sequence $\{x_n\}$ of rational numbers such that $\lim_{n\to\infty} x_n = x$. Hence, it holds that

$$f(a^{x_n}) = x_n f(a). \qquad (**)$$

where we use the conclusion of (b).

Since $f(x)$ is a continuous function, we have

$$\begin{cases} \lim_{n\to+\infty} f(a^{x_n}) = f(\lim_{n\to+\infty} a^{x_n}) = f(a^x), \\ \lim_{n\to\infty} x_n f(a) = x \cdot f(a), \end{cases}$$

By (**), it holds that $f(a^x) = xf(a)$, for all $x \in \mathbb{R}$.

Finally, let $x = a^y$. We have $y \cdot \log a = \log x$. Hence,

$$f(x) = f(a^y) = y \cdot f(a) = \frac{f(a)}{\log a} \log x.$$

By replacing $\frac{f(a)}{\log a}$ by c, it holds that $f(x) = c\log x$. $\square$