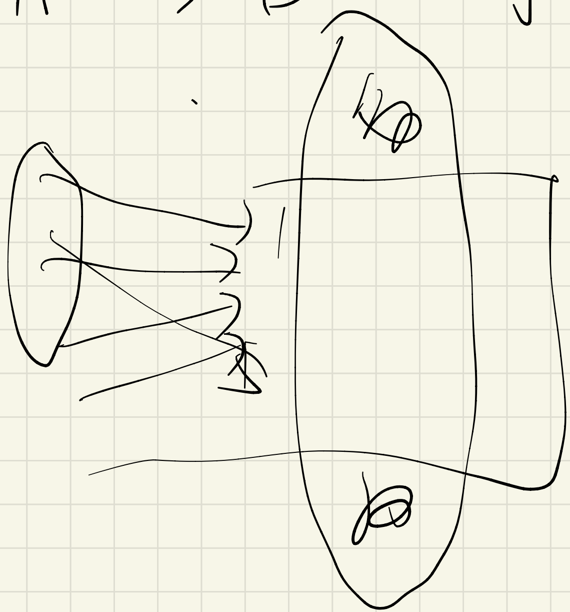
Functions: Domain, Image, composition, injective/surjective, inverse

piecewise constant functions

$f: A \to B \qquad y = f(x)$



A: Domain

$\text{Image} = f(A)$

$= \{ y \in B :$ there is an x s.t. $f(x) = y \}$

$$A \xrightarrow{f} B$$

$$x \longmapsto y$$

injective if

$$\underline{f(x_1) = f(x_2)} \Rightarrow x_1 = x_2$$

surjective if

for any $y \in B$, there is an x.

$$f(x) = y$$

bijective if both

injective & surjective

$\angle$

inverse: $A \xrightarrow{f} B$

inverse is a map

$B \xrightarrow{g} A$

such that:

$f \circ g(x) = g \circ f(x) = x$

$A \xrightarrow{f} B \xrightarrow{g} A$

$x \qquad\qquad g \circ f \qquad\qquad x$

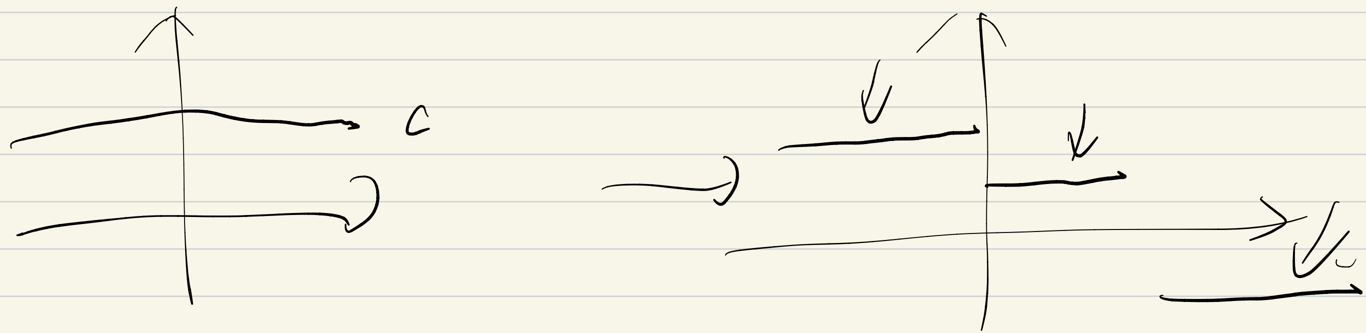
$f \circ g = id_B$

$g \circ f = id_A$

$B \xrightarrow{g} A \xrightarrow{f} B \rightarrow f \circ g$

$x \qquad\qquad\qquad x$

piece wise constant :



(2) Given the functions $f(x) = \frac{x-4}{x-6}$ and $g(x) = \sqrt{x+3}$, find the domains of f, g, $f+g$, $\frac{f}{g}$, $\frac{g}{f}$, $f \circ g$, $g \circ f$.

$$f(x) = \frac{x-4}{x-6} \longrightarrow x \neq 6$$

Domain of f: $(-\infty, 6) \cup (6, +\infty)$

$$g(x) = \sqrt{x+3} \Longrightarrow x \geq -3$$

g: $[-3, +\infty)$

$$(f+g)(x) = \frac{x-4}{x-6} + \sqrt{x+3}$$

$$x \neq 6 \quad , \quad x \geq -3$$

$$f+g: \ [-3, 6) \cup (6, +\infty)$$

$$f \circ g(x) = f(g(x)) = f(\sqrt{x+3}) = \frac{\sqrt{x+3} - 4}{\sqrt{x+3} - 6}$$

$$x \geq -3 \qquad \sqrt{x+3} - 6 \neq 0 \iff x \neq 33$$

$$[-3, 33) \cup (33, +\infty)$$

(3) Suppose $f(x) = 6x - 9$ and $g(y) = \frac{y}{6} + \frac{9}{6}$.

(a) Find the composition $g(f(x))$.

(b) Find the composition $f(g(y))$.

(c) Are the functions f and g inverse to each other?

(i) $(g \circ f)(x) = g(f(x)) = g(6x - 9) = \frac{6x-9}{6} + \frac{9}{6}$

$= x$

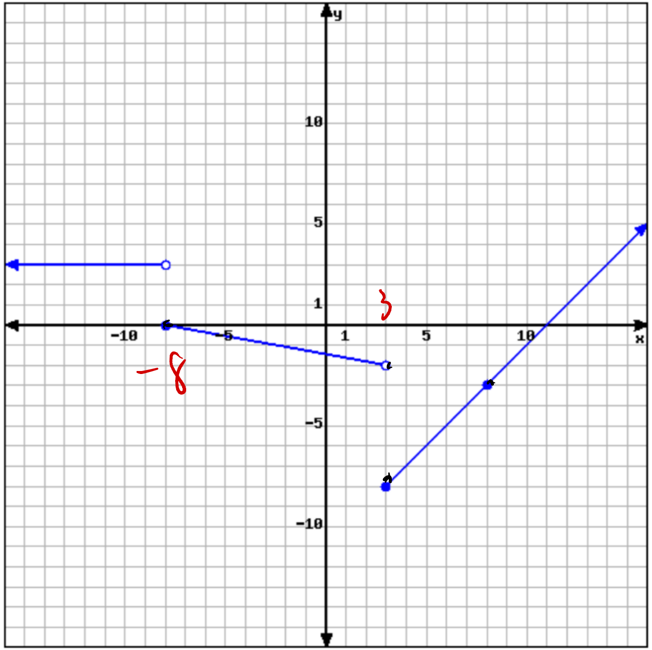
(ii) $(f \circ g)(y) = f(g(y)) = f\left(\frac{y}{6} + \frac{9}{6}\right)$

$= 6 \cdot \left(\frac{y}{6} + \frac{9}{6}\right) - 9$

$= y$

(iii) ✓

(5) The graph is a piecewise function, $f(x)$, is depicted below. Find its equation.



$x < -8: \ f(x) = 3$

$-8 \leq x < 3$

$f(x) = ax + b$

$f(8) = 0$

$f(3^-) = -2$

$\Rightarrow \begin{cases} -8a + b = 0 \\ 3a + b = -2 \end{cases}$

$\Rightarrow \begin{cases} a = -\frac{2}{11} \\ b = -\frac{16}{11} \end{cases}$

$x \geq 3$

$f(x) = cx + d$

$f(3^+) =$

$f(8) =$

Exercise 7. (level 2)

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = |x-1| - |x+1|$ for any $x \in \mathbb{R}$.

(a) Express the 'explicit formula' of the function f as that of a piecewise defined function, with one 'piece' for $(-\infty, -1)$, $[-1, 1]$, $(1, +\infty)$.

(b) Sketch the graph of the function f.

(c) Is f an injective function on $\mathbb{R}$? Justify your answer.

(d) What is the image of $\mathbb{R}$ under the function f?

(a) $|x-1| :$ $x \geq 1$ $x-1$

$x < 1$ $1-x$

$|x+1| :$ $x \geq -1$ $x+1$

$x < -1$ $-(x+1)$

$x < -1$, $-1 \leq x < 1$, $x \geq 1$

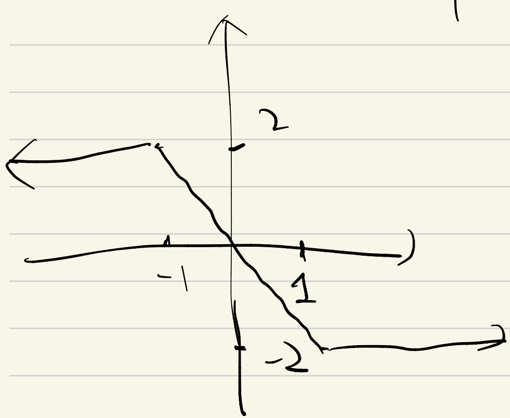
$f(x) = 1-x - (-(x+1)) = 2$

$-1 \leq x < 1$

$f(x) = 1 - x - (x+1) = -2x$

$x > 1 \quad f(x) = x - 1 - (x+1) = -2$

$$f(x) = \begin{cases} 2 , & x < -1 \\ -2x , & -1 \leq x < 1 \\ -2 , & x \geq 1 \end{cases}$$



(3) X

(4) Image : $[-2, 2]$

Exercise 8. (Level 2)

(a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^3$ for any $x \in \mathbb{R}$. Verify that f is injective on $\mathbb{R}$.

(b) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $g(x) = x^3 + x$ for any $x \in \mathbb{R}$. Verify that g is injective on $\mathbb{R}$.

(c) Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $h(x) = x^3 - x$ for any $x \in \mathbb{R}$. Verify that h is not injective on $\mathbb{R}$.

(a) if $f(x_1) = f(x_2)$ $\quad x_1^3 = x_2^3$

$\Rightarrow x_1^3 - x_2^3 = 0$, $\quad (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0$

$$x_1^2 + x_1x_2 + x_2^2 = \left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2$$

$$\overset{\shortparallel}{0}$$

$x_2 = 0 \quad x_1 + \frac{x_2}{2} = 0$

$\Rightarrow x_1 = x_2 = 0$

$\Rightarrow x_1 = x_2$

(b) If $g(x_1) = g(x_2)$ or $x_1^3 + x_1 = x_2^3 + x_2$

$$\Rightarrow \quad x_1^3 - x_2^3 + (x_1 - x_2) = 0$$

$$\Rightarrow \quad (x_1 - x_2)(x_1^2 + x_1 x_2 + x_2^2 + 1) = 0$$

$$\left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2 + 1 \neq 0$$

$$\Rightarrow \quad x_1 - x_2 = 0$$

(c) $h(x) = x^3 - x$ is not injective

want to find: $x_1 \neq x_2$, to make

$$h(x_1) = h(x_2)$$

$$h(x) = x(\underbrace{x^2 - 1})$$

$$x_1 = 1 \qquad x_2 = -1$$

$$f(x) = \underbrace{1 + x - 2x^2 + 3x^3 - 4x^4}_{\mathbb{R}}$$

$\forall y \in \mathbb{R}$ find an x,

$$f(x) = y$$