2025 Fall Real Analysis I
Midterm Examination

Answer four questions. Do not do all five. Notations in Lecture Notes are in effect.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) Let $A, B \in \mathcal{M}$. Show that

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

Solution:

$$\mu(A \cup B) = \mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A) = \mu(A) + \mu(B) - \mu(A \cap B).$$

(b) Let $A_k \in \mathcal{M}, k \geq 1$. Assume that $\sum_{k=1}^{\infty} \mu(A_k) < \infty$. Show that the set

$$E = \{x \in X : x \text{ belongs to infinitely many } A_k\},$$

is a null set. You may assume E to be measurable.

Solution: Since $\sum_{k=1}^{\infty} \mu(A_k) < \infty$, we have $\sum_{k=n}^{\infty} \mu(A_k) \to 0$ as $n \to \infty$. For any $n \in \mathbb{N}$, we have

$$E \subset \bigcup_{k \geq n} A_k$$

and so

$$\mu(E) \leq \sum_{k=n}^{\infty} \mu(A_k).$$

Taking $n \to \infty$, we have $\mu(E) = 0$.

This result is called Borel–Cantelli lemma.

2. (a) Let f, g be two measurable functions on a measurable space $(X, \mathcal{M})$. Show that the sum $f + g$ and the product $h = fg$ are measurable.

Solution: It suffices to show

$$(f+g)^{-1}(a,\infty) = \bigcup_{\substack{t+s>a \\ t,s\in\mathbb{Q}}} f^{-1}(t,\infty) \cap g^{-1}(s,\infty).$$

Let's consider a point x satisfying $(f+g)(x) > a$. We can always choose two rational numbers t and s such that $f(x) > t, g(x) > s$ and $t + s > a$. It follows that $x \in f^{-1}(t,\infty) \cap g^{-1}(s,\infty)$, and we have one side inclusion.

From

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right],$$

it suffices to prove that f^2 is measurable whenever f is measurable. For any $a \in \mathbb{R}$:

- If $a < 0$, then $\{x : f^2(x) > a\} = X$ is measurable;
- If $a \geq 0$, then

  $$\{x : f^2(x) > a\} = \{x : f(x) > \sqrt{a}\} \cup \{x : f(x) < -\sqrt{a}\}$$

  which is measurable since f is measurable.

Therefore, f^2 is measurable.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and injective. Show that $f(B) \in \mathcal{B}$ for any $B \in \mathcal{B}$, where $\mathcal{B}$ stands for the Borel σ-algebra in $\mathbb{R}$.

Solution: Let $\mathcal{F} := \{B \in \mathcal{B} : f(B) \in \mathcal{B}\}$. We first show that $\mathcal{F}$ is a σ-algebra:

- Since $f(\mathbb{R}) = \mathbb{R} \in \mathcal{B}$, we have $\mathbb{R} \in \mathcal{F}$;
- If $B \in \mathcal{F}$, then $f(B) \in \mathcal{B}$, whence $f(\mathbb{R} \setminus E) = \mathbb{R} \setminus f(B) \in \mathcal{B}$, which shows $\mathbb{R} \setminus B \in \mathcal{F}$;
- If $E_i \in \mathcal{F}$, then $f(E_i) \in \mathcal{B}$ for all i, whence $f(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} f(E_i) \in \mathcal{B}$, which shows $\bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$.

Since f is continuous, $\mathcal{F}$ contains all compact sets in $\mathbb{R}$. As $\mathcal{B}$ is the smallest σ-algebra containing all open sets in $\mathbb{R}$, we have $\mathcal{B} \subseteq \mathcal{F}$. Consequently, for all $B \in \mathcal{B}$, we have $f(B) \in \mathcal{B}$.

3. (a) State Lusin's theorem without proof.

Solution: Let μ be a Riesz measure on a locally compact Hausdorff space X and let f be a real-valued measurable function vanishing outside some measurable set A, $\mu(A) < \infty$. For every $\varepsilon > 0$, there exists some $g \in C_c(X)$, such that

$$\mu\big(\{x : \ f(x) \neq g(x)\}\big) < \varepsilon.$$

Moreover, when f is bounded, g can be chosen to satisfy

$$\sup_{x\in X} |g(x)| \leq \sup_{x\in X} |f(x)|.$$

(b) Let f be an integrable function on a measure space $(X, \mathcal{M}, \mu)$. Show that for each $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\int_E |f| \, d\mu < \epsilon$$

for each $E \in \mathcal{M}$ with $\mu(E) < \delta$.

Solution: Assume on the contrary there is some $\varepsilon_0 > 0$ and $E_n, \mu(E_n) \leq 2^{-n}$, such that $\int_{E_n} |f| d\mu \geq \varepsilon_0$. Let $A_n = \bigcup_{j\geq n} E_j$. Then

$$\mu(A_n) \leq \sum_{j\geq n} \mu(E_j) \leq \sum_{j\geq n} \frac{1}{2^j} = \frac{1}{2^{n-1}}.$$

Let $A = \cap_n A_n$. As $\{A_n\}$ is descending and $\mu(A_1)$ is finite,

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) = 0,$$

that is, A is of measure zero. On the other hand, we have $|f|\chi_{A_n} \leq |f|$, by the dominated convergence theorem we have

$$\int_A |f| d\mu = \lim_{n\to\infty} \int_{A_n} |f| d\mu \geq \varepsilon_0 > 0,$$

contradiction holds.

4. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$.

(a) Suppose that Φ is continuous on $\mathbb{R}$. Is it true that $\Phi(E)$ is always Lebesgue measurable? Prove this or give a counter-example (and prove that it is a counter-example).

Solution: The answer is no. To construct a counter example, let $h : [0,1] \to [0,2]$ be the function given by lecture notes Ch3 section 3.2, i.e. $h(x) := x + g(x)$ where g is the Cantor function. Define $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(x) := \begin{cases} x & \text{if } x < 0 \\ h(x) & \text{if } 0 \leq x \leq 1 \\ x+1 & \text{if } 1 < x. \end{cases}$$

Using the property of h, we see that Φ is an injective and continuous function on $\mathbb{R}$. Denoting the Cantor set by $\mathcal{C}$, we have $\mathcal{L}(\Phi(\mathcal{C})) = \mathcal{L}(h(\mathcal{C})) = 1$ by the property of h. Therefore, by lecture notes Ch3 Proposition 3.3, there exists some non-measurable $A \subseteq \Phi(\mathcal{C})$. Since Φ is injective, $E := \Phi^{-1}(A)$ is a subset of $\mathcal{C}$. As $\mathcal{C}$ is of measure zero, E is a measurable set, while $\Phi(E) = A$ is not measurable.

(b) Suppose that

$$|\Phi(y) - \Phi(x)| \leq L|x-y|, \quad \forall\, x, y \in E,$$

for some positive constant L. Show that $\Phi(E)$ is Lebesgue measurable.

Solution: Assume that E is compact first. As the image of a compact set under a continuous map is again compact and so is Borel, we see that $\Phi(E)$ is also compact, hence measurable. Next, let E be a bounded measurable set. By inner regularity we can find a set $F \subset E$ which is the countable union of compact sets satisfying $\mathcal{L}(E \setminus F) = 0$. Hence the set $N = E \setminus F$ is null and $\Phi(E) = \Phi(F) \cup \Phi(N)$. We have $\Phi(F) = \bigcup_j \Phi(K_j)$ where K_j are compact, so $\Phi(F)$ is Borel (hence measurable). Therefore, things boil down to show that the image of a null set under a Lipschitz map is a null set. This is the key point, and the proof is not difficult. Finally, we can write a measurable set as the countable union of bounded, measurable sets.

5. Let μ_1 and μ_2 be two measures on a measurable space $(X, \mathcal{M})$. Define

$$\mu(E) = \inf\{\mu_1(E \cap F) + \mu_2(E \setminus F) : \; F \in \mathcal{M}\}$$

for $E \in \mathcal{M}$. Prove that μ is a measure on $(X, \mathcal{M})$.

Solution: Plainly μ is a nonnegative function on $\mathcal{M}$ and $\mu(\emptyset) = 0$. Let $\{E_k\}$ be a countable collection of mutually disjoint sets in $\mathcal{M}$. Writing $E := \bigcup_k E_k$, we would like to show that

$$\mu(E) = \sum_k \mu(E_k).$$

On the one hand, given $F_0 \in \mathcal{M}$, we have

$$\begin{aligned}\sum_k \mu(E_k) &= \sum_k \inf\{\mu_1(E_k \setminus F) + \mu_2(E_k \cap F) : F \in \mathcal{M}\} \\ &\leq \sum_k [\mu_1(E_k \setminus F_0) + \mu_2(E_k \cap F_0)] \\ &= \mu_1(E \setminus F_0) + \mu_2(E \cap F_0),\end{aligned}$$

whence $\sum_k \mu(E_k) \leq \mu(E)$ by taking inf over $F_0 \in \mathcal{M}$ on the R.H.S.

To get the reverse inequality, let $\varepsilon > 0$ be fixed. For each k, there exists $F_k \in \mathcal{M}$ such that

$$\mu_1(E_k \setminus F_k) + \mu_2(E_k \cap F_k) \leq \mu(E_k) + \frac{\varepsilon}{2^k}.$$

Let $F := \bigcup_k (E_k \cap F_k)$. Note that $F \subseteq E$ and $E \setminus F = \bigcup_k (E_k \setminus F_k)$. Hence

$$\begin{aligned}\mu(E) &\leq \mu_1(E \setminus F) + \mu_2(E \cap F) \\ &= \sum_k \mu_1(E_k \setminus F_k) + \sum_k \mu_2(E_k \cap F_k) \\ &= \sum_k [\mu_1(E_k \setminus F_k) + \mu_2(E_k \cap F_k)] \\ &\leq \sum_k \mu(E_k) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we finish the proof.