Real Analysis 25-11-28

§5.2 Radon-Nikodym Thm

Def (Absolute continuity).

Let μ be a measure on $(X, \mathcal{M})$.

Let λ be a measure / signed measure on $(X, \mathcal{M})$.

We say that λ is abs. cts w.r.t μ if

$E \in \mathcal{M}$, $\mu(E) = 0 \implies \lambda(E) = 0$.

We write $\lambda \ll \mu$.

Def.
- Say that λ is concentrated on $A \in \mathcal{M}$ if $\lambda(E) = \lambda(E \cap A) \quad \forall\ E \in \mathcal{M}$.
- Say that $\lambda_1 \perp \lambda_2$ if $\exists$ disjoint $A, B \in \mathcal{M}$ such that λ_1 is concentrated on A, λ_2 is concentrated on B.

Prop 5.3. Let μ be a measure on $(X, \mathcal{M})$

Let λ_1, λ_2 be measures / signed measures on $(X, \mathcal{M})$.

Then the following hold:

(a) λ is concentrated on $A \in \mathcal{M}$

$\Rightarrow$ $|\lambda|$ is concentrated on $A \in \mathcal{M}$

(b) $\lambda_1 \perp \lambda_2 \Rightarrow |\lambda_1| \perp |\lambda_2|$

(c) $\lambda_1 \perp \mu, \ \lambda_2 \perp \mu \Rightarrow \lambda_1 + \lambda_2 \perp \mu$

(d) $\lambda_1 \ll \mu, \ \lambda_2 \ll \mu \Rightarrow \lambda_1 + \lambda_2 \ll \mu.$

(e) $\lambda \ll \mu \Rightarrow |\lambda| \ll \mu$

(f) $\lambda_1 \ll \mu, \ \lambda_2 \perp \mu \Rightarrow \lambda_1 \perp \lambda_2$

(g) $\lambda \ll \mu, \ \lambda \perp \mu \Rightarrow \lambda = 0.$

Pf of (g): Since $\lambda \perp \mu$, $\exists$ disjoint $A, B \in \mathcal{M}$ for which λ is concentrated on A, μ is concentrated on B.

Hence $\forall \ E \in \mathcal{M}$,

$$\lambda(E) = \lambda(E \cap A).$$

In the mean time,

$$\mu(E \cap A) = \mu(E \cap A \cap B) = 0.$$

Since $\lambda \ll \mu$, $\lambda(E \cap A) = 0$

Hence $\lambda(E) = \lambda(E \cap A) = 0.$ $\blacksquare$

Thm 5.5 (Lebesgue decomposition)

Let μ be a σ-finite measure on $(X, \mathcal{M})$.

Let λ be a signed measure on $(X, \mathcal{M})$.

Then $\exists$ a unique decomposition

$$\lambda = \lambda_{ac} + \lambda_s$$

Such that $\lambda_{ac} \ll \mu$ and $\lambda_s \perp \mu$.

Thm 5.6 (Radon - Nikodym Thm).

Under the same assumptions in Thm 5.5.

Suppose that $\lambda \ll \mu$.

Then $\exists$ a unique $h \in L^1(\mu)$ such that

$$\lambda(E) = \int_E h \, d\mu, \quad \forall E \in \mathcal{M}.$$

Pf of Thm 5.5 & Thm 5.6.

Step 1. We first consider the case when both μ and λ are finite measures.

Let $\rho = \mu + \lambda$. Then ρ is a finite measure. Next we introduce $\Lambda : L^2(\rho) \to \mathbb{R}$ by

$$① \quad \Lambda(\phi) = \int \phi \, d\lambda, \qquad \forall \phi \in L^2(\rho).$$

By Cauchy-Schwartz inequality,

$$\begin{aligned} |\Lambda(\phi)| &= \left| \int \phi \, d\lambda \right| \\ &\leq \left(\int \phi^2 d\lambda \right)^{1/2} \left(\int 1 \, d\lambda \right)^{1/2} \\ &\leq \left(\int \phi^2 d\rho \right)^{1/2} \lambda(X)^{1/2} \end{aligned}$$

$$= \|\phi\|_{L^2(\rho)} \cdot \lambda(X)^{1/2}$$

Hence $\Lambda \in L^2(\rho)'$

Hence $\exists\ g \in L^2(\rho)$ such that

$$(2) \qquad \Lambda(\phi) = \int \phi g \, d\rho, \qquad \forall\ \phi \in L^2(\rho).$$

$$= \int \phi \, d\lambda$$

Taking $\phi = \chi_E$ in (2) gives

$$\Lambda(\chi_E) = \int \chi_E \, g \, d\rho = \int_E g \, d\rho$$

$$= \int \chi_E \, d\lambda = \lambda(E).$$

That is,

$$\lambda(E) = \int_E g \, d\rho, \qquad \forall\ E \in \mathcal{M}$$

So

$$0 \leq \frac{1}{\rho(E)} \int_E g \, d\rho = \frac{\lambda(E)}{\rho(E)} \leq 1, \qquad \forall E \in \mathcal{M}.$$

This implies $0 \leq g(x) \leq 1$ for ρ-a.e x.

Suppose on the contrary that $\exists\ \varepsilon > 0$ such that

either ① $g > 1+\varepsilon$ on a set C with $\rho(C) > 0$.

or ② $g < -\varepsilon$ on a set $\tilde{C}$ with $\rho(\tilde{C}) > 0$.

If ① holds,

$$\frac{1}{\rho(C)} \int_C g \, d\rho \geq 1+\varepsilon$$ leading to a contradiction

If ② holds

$$\frac{1}{\rho(\tilde{C})} \int_{\tilde{C}} g \, d\rho \leq -\varepsilon$$, a contradiction

Now by redefining g on a null set we may assume

$$0 \leq g(x) \leq 1, \quad \forall\ x \in X.$$

Then we define

$$A = \{x : \ g(x) \in [0,1)\},$$
$$B = \{x : \ g(x) = 1\}.$$

Define for $E \in \mathcal{M}$,

$$\lambda_{ac}(E) = \lambda(E \cap A)$$

$$\lambda_s(E) = \lambda(E \cap B).$$

By definition, λ_{ac} is concentrated on A

λ_s is concentrated on B.

Recall that

$$\int \phi \, d\lambda = \int \phi g \, d\rho, \quad \forall \phi \in L^2(\rho)$$

$$= \int \phi g \, d\lambda + \int \phi g \, d\mu$$

We have

$$③ \int \phi(1-g) \, d\lambda = \int \phi g \, d\mu.$$

Taking $\phi = \chi_B$ in ③ gives

$$\int_B 1-g \, d\lambda = \int_B g \, d\mu$$

Hence $0 = \mu(B)$.

Recall that λ_s is concentrated on B,

but μ is concentrated on $X \setminus B$

So $\lambda_s \perp \mu$.

Next we prove $\lambda_{ac} \ll \mu$.

To see this, taking $\phi = \chi_E (1+g+\cdots+g^n)$ in (3)

gives

$$\int \chi_E (1+g+\cdots+g^n)(1-g)\, d\lambda = \int \chi_E (1+g+\cdots+g^n)\, g\, d\mu$$

$$\begin{aligned}
LHS &= \left(\int_A + \int_B\right) \chi_E (1+g+\cdots+g^n)(1-g)\, d\lambda \\
&= \int_{A\cap E} (1+g+\cdots+g^n)(1-g)\, d\lambda \\
&= \int_{A\cap E} 1 - g^{n+1}\, d\lambda \\
&\longrightarrow \int_{A\cap E} 1\, d\lambda = \lambda(E\cap A) = \lambda_{ac}(E)
\end{aligned}$$

as $n \to \infty$.

$$(\text{RHS}) = \left(\int_A + \int_B\right) \chi_E (1+g+\cdots+g^n) g \, d\mu$$

$$\overset{\mu(B)=0}{=\!=} \int_{A\cap E} g \cdot \frac{1-g^{n+1}}{1-g} \, d\mu$$

$$\longrightarrow \int_{A\cap E} \frac{g}{1-g} \, d\mu \qquad \text{as } n \to \infty$$

$$= \int_E \frac{g}{1-g} \, d\mu \qquad (\text{since } \mu(B)=0)$$

Hence

$$\lambda_{ac}(E) = \int_E \frac{g}{1-g} \, d\mu, \quad \forall \; E \in \mathcal{M}.$$

set $h = \dfrac{g}{1-g}$. Then

$$\lambda_{ac}(E) = \int_E h \, d\mu$$

which implies $\infty > \lambda_{ac}(X) = \int_X h \, d\mu$,

Hence $h \in L^1(\mu)$ and

$$\lambda_{ac}(E) = \int_E h \, d\mu \Rightarrow \lambda_{ac} \ll \mu.$$

Step 2. Assume μ is finite, λ is a signed measure.

Define

$$\lambda^+ = \frac{1}{2}(\lambda + |\lambda|),$$

$$\lambda^- = \frac{1}{2}(|\lambda| - \lambda).$$

Then from the property that

$$|\lambda(E)| \leq |\lambda|(E), \quad E \in \mathcal{M}$$

We see that

λ^+, λ^- are two finite measures on $(X, \mathcal{M})$.

Moreover $\lambda = \lambda^+ - \lambda^-$.

We call the above decomposition the John decomposition of λ.

By step 1,

$$\lambda^+ = \lambda^+_{ac} + \lambda^+_s$$

$$\lambda^- = \lambda^-_{ac} + \lambda^-_s$$

where $\lambda^+_{ac} \ll \mu$, $\lambda^-_{ac} \ll \mu$, $\lambda^+_s, \lambda^-_s \perp \mu$.

Hence

$$\lambda = \lambda^+ - \lambda^-$$

$$= (\lambda_{ac}^+ - \lambda_{ac}^-) + (\lambda_s^+ - \lambda_s^-)$$

Now $\lambda_{ac}^+ - \lambda_{ac}^- \ll \mu$, $\lambda_s^+ - \lambda_s^- \perp \mu$.

Step 3. Now consider the case that μ is σ-finite and λ is a signed measure.

Let $\{X_j\}_{j=1}^{\infty}$ be a partition of X such that

$\mu(X_j) < \infty$.

Write

$$\mu_j = \mu|_{X_j}$$

$$\lambda_j = \lambda|_{X_j}$$

(i.e. $\mu_j(E) = \mu(E \cap X_j)$, $\lambda_j(E) = \lambda_j(E \cap X_j)$)

Then μ_j are finite measures on $(X, \mathcal{M})$

λ_j are signed measures on $(X, \mathcal{M})$.

Then $\lambda_j = \lambda_{ac}^j + \lambda_s^j$

$$\lambda_{ac}^j \ll \mu_j$$

$$\lambda_s^j \perp \mu_j$$

Then $\exists\ h_j \in L^1(\mu_j)$ s.t

$$\lambda_{ac}^j(E) = \int_E h_j \, d\mu_j, \quad j = 1, \cdots$$

Finally let

$$\lambda_{ac} = \sum_{j=1}^{\infty} \lambda_{ac}^j$$

$$\lambda_s = \sum_{j=1}^{\infty} \lambda_s^j$$

Then $\lambda_{ac} \ll \mu$ and $\lambda_s \perp \mu$

$$\lambda_{ac}(E) = \int_E h \, d\mu,$$

where

$$h = \sum_{j=1}^{\infty} h_j \chi_{X_j} .$$

Step 4. (Uniqueness of the Leb. decomposition)

μ —— σ-finite measure

λ —— a signed measure.

Suppose

$$\lambda = \lambda_1 + \lambda_2$$
$$= \lambda_3 + \lambda_4$$

Such that $\lambda_1, \lambda_3 \ll \mu$

$\lambda_2, \lambda_4 \perp \mu$.

Notice that $\lambda_1 - \lambda_3 \ll \mu$

But $\lambda_1 - \lambda_3 = \lambda_4 - \lambda_2 \perp \mu$

By (g) of Prop 5.3, $\lambda_1 - \lambda_3 = 0$

Hence $\lambda_1 = \lambda_3$, $\lambda_2 = \lambda_4$. $\blacksquare$

Uniqueness part in R-N Thm.

Suppose $\lambda \ll \mu$ and $\exists$ $h_1, h_2 \in L^1(\mu)$ such that

$$\lambda(E) = \int_E h_1 \, d\mu = \int_E h_2 \, d\mu, \quad \forall E \in \mathcal{M}$$

Hence

$$\int_E h_1 - h_2 \, d\mu = 0, \quad \forall E \in \mathcal{M}.$$

We need to show that $h_1 - h_2 = 0$ a.e.

If not, then $\exists$ $\varepsilon > 0$ such that either

$h_1 - h_2 > \varepsilon$ on a set B of positive measure

or $h_1 - h_2 < -\varepsilon$ on a set B of positive meas.

$$\int_B h_1 - h_2 \, d\mu \geq \varepsilon \, \mu(B) > 0$$

leading a contradiction.

Prop 5.7. Let μ be a signed measure on $(X, \mathcal{M})$.

Let $|\mu|$ denote the total variation of μ.

Then the following hold:

① $\exists \; h \in L^1(|\mu|)$ such that $|h| = 1$ for $|\mu|$-a.e.

and

$$\mu(E) = \int_E h \, d|\mu|.$$

② $\exists$ disjoint $A, B \in \mathcal{M}$ such that

$$\mu^+(E) = \mu(E \cap A), \quad \forall E \in \mathcal{M}$$

$$\mu^-(E) = -\mu(E \cap B), \quad \forall E \in \mathcal{M},$$

where $\mu^+ = \frac{1}{2}(|\mu| + \mu)$, $\mu^- = \frac{1}{2}(|\mu| - \mu)$.

③ If $\mu = \lambda_1 - \lambda_2$ for two measures λ_1, λ_2

then $\lambda_1 \geq \mu^+$, $\lambda_2 \geq \mu^-$.

Pf. Since $|\mu(E)| \leq |\mu|(E)$, $\mu \ll |\mu|$.

By the Radon-Nikodym Thm,

$$\exists \quad h \in L^1(|\mu|) \quad \text{such that}$$

$$\mu(E) = \int_E h \, d|\mu|, \quad \forall E \in \mathcal{M}.$$

First we prove that $|h| \leq 1$ for $|\mu|$-a.e.

If not, then $\exists \ \varepsilon > 0$ such that

either $h > 1 + \varepsilon$ on a set E with $|\mu|(E) > 0$

or $h < -(1+\varepsilon)$ on a set E with $|\mu|(E) > 0$.

WLOG, suppose the first case occurs.

$$\mu(E) = \int_E h \, d|\mu|(E) \geq (1+\varepsilon) \underbrace{|\mu|(E)}_{>0},$$

which is impossible, because $|\mu(E)| \leq |\mu|(E)$.

Next we show that $|h| \geq 1$ for $|\mu|$-a.e.

For $0 < r < 1$, define

$$A_r = \{ x : \ |h(x)| < r \}.$$

Then if $\{E_j\}$ is a partition of A_r,

then
$$|\mu(E_j)| = \left| \int_{E_j} h \, d|\mu| \right|$$
$$\leq \int_{E_j} |h| \, d|\mu|$$
$$\leq \int_{E_j} r \, d|\mu|$$
$$\leq r \cdot |\mu|(E_j).$$

Hence
$$\sum_{j=1}^{\infty} |\mu(E_j)| \leq r \cdot \sum_{j=1}^{\infty} |\mu|(E_j)$$
$$= r \cdot |\mu|(A_r)$$

Taking supremum over $\{E_j\}$ gives

$$|\mu|(A_r) \leq r \cdot |\mu|(A_r)$$

$$\Rightarrow \quad |\mu|(A_r) = 0.$$

So

$$|\mu|\left\{ x : |h| < 1 \right\}$$

$$= |\mu|\left(\bigcup_{n=1}^{\infty} A_{\frac{n}{n+1}} \right) \leq \sum_{n=1}^{\infty} |\mu|\left(A_{\frac{n}{n+1}}\right)$$

$$= 0.$$

Hence $|h| \geq 1$ $\quad |\mu|$-a.e.

So $|h| = 1$ $\quad |\mu|$-a.e.

This completes the proof of 1.

Next set

$$A = \{ x : h(x) = 1 \}$$

$$B = \{ x : h(x) = -1 \}.$$

Then

$$\mu(E \cap A) = \int_{E \cap A} h \, d|\mu|$$

$$= \int_{E \cap A} 1 \, d|\mu|$$

$$= |\mu|(E \cap A), \quad (1)$$

$$\mu(E \cap B) = \int_{E \cap B} h \, d|\mu|$$

$$= \int_{E \cap B} (-1) \, d|\mu|$$

$$= -|\mu|(E \cap B) \quad (2)$$

From this, we can check that

$$\mu^+ = \mu|_A \quad \text{and} \quad \mu^- = -\mu|_B$$

Indeed $\mu^+(E) = \frac{1}{2}\left(|\mu|(E) + \mu(E)\right)$

$$= \frac{1}{2}\left(|\mu|(E\cap A) + |\mu|(E\cap B) + \mu(E\cap A) + \mu(E\cap B)\right)$$

$$= \frac{1}{2}\left(|\mu|(E\cap A) + \mu(E\cap A)\right)$$

$$= \mu(E\cap A).$$

Similarly $\mu^-(E) = -\mu(E\cap B)$.

3. Suppose

$\mu = \lambda_1 - \lambda_2$, where λ_1, λ_2 are measures.

We need to show that

$$\lambda_1 \geq \mu^+$$

$$\lambda_2 \geq \mu^-.$$

Notice by 2, $\exists$ A, B disjoint, such that

$$\mu^+(E) = \mu(E\cap A)$$

$$\mu^-(E) = -\mu(E\cap B).$$

So

$$\begin{aligned}\mu^+(E) &= \mu(E \cap A) \\ &= \lambda_1(E \cap A) - \lambda_2(E \cap A) \\ &\leq \lambda_1(E \cap A) \\ &\leq \lambda_1(E).\end{aligned}$$

$$\begin{aligned}\mu^-(E) &= -\mu(E \cap B) \\ &= \lambda_2(E \cap B) - \lambda_1(E \cap B) \\ &\leq \lambda_2(E \cap B) \\ &\leq \lambda_2(E).\end{aligned}$$

$\blacksquare$