

Real Analysis

25-11-21

• Vitali convergence Thm.

Def. (Unif. Integrability)

Let $(f_n) \subset L^1(X, M, \mu)$. We say (f_n) is unif. integrable if

$$\textcircled{1} \sup_n \int |f_n| d\mu < \infty.$$

$\textcircled{2} \forall \varepsilon > 0, \exists \delta > 0$ such that

$$\int_E |f_n| d\mu < \varepsilon \text{ if } E \in M, \mu(E) < \delta, n \in \mathbb{N}.$$

Thm 4.27 (Vitali convergence Thm)

Let $\mu(X) < \infty$ and $f_n \in L^1(\mu), n \geq 1$. Assume

$$\textcircled{1} f_n \rightarrow f \text{ a.e.}$$

$\textcircled{2} (f_n)$ is uniformly integrable.

Then $f_n \rightarrow f$ in $L^1(\mu)$, i.e. $\int |f_n - f| d\mu \rightarrow 0$.

Proof. Let $\varepsilon > 0$. $\exists \delta > 0$ such that

$$\sup_n \int_E |f_n| d\mu < \varepsilon \quad \text{if} \quad \mu(E) < \delta. \quad (1)$$

By Fatou lemma, if $\mu(E) < \delta$, then

$$\begin{aligned} \int_E |f| d\mu &= \int_E \lim_{n \rightarrow \infty} |f_n| d\mu \\ &\leq \lim_{n \rightarrow \infty} \int_E |f_n| d\mu \\ &< \varepsilon. \end{aligned} \quad (2)$$

Since $\mu(X) < \infty$, $f_n \rightarrow f$ a.e., by Egorov's Thm,

$\exists A \in \mathcal{M}$ with $\mu(A) < \delta$ such that

$$f_n \rightrightarrows f \quad \text{on} \quad X \setminus A.$$

Hence $\exists N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \text{if} \quad x \in X \setminus A, n \geq N.$$

Now for $n \geq N$,

$$\begin{aligned}\int |f_n - f| d\mu &= \int_{X \setminus A} |f_n - f| d\mu + \int_A |f_n - f| d\mu \\ &\leq \varepsilon \cdot \mu(X \setminus A) + \int_A |f_n| d\mu \\ &\quad + \int_A |f| d\mu \\ &\leq \varepsilon \cdot \mu(X) + 2\varepsilon \quad (\text{by (1) and (2)})\end{aligned}$$

Hence we obtain the desired result.



Chap 5. Radon-Nikodym Thm.

§ 5.1 Signed measures (符号测度)

Def. Let (X, \mathcal{M}) be a measurable space.

A function $\mu: \mathcal{M} \rightarrow \mathbb{R}$ is said to be a signed measure if

$$\mu(E) = \sum_{n=1}^{\infty} \mu(E_n) \quad \text{if } (E_n) \text{ is a partition of } E, \quad \forall E \in \mathcal{M}.$$

(i.e. $E_n \in \mathcal{M}, n \geq 1$ are disjoint subsets of E
with $\bigcup_{n=1}^{\infty} E_n = E$)

Remark: (1) It is clear that $\mu(\emptyset) = 0$
also $|\mu(X)| < \infty$.

Hence a measure μ may be not a signed measure.

Def. Given a signed measure μ on (X, \mathcal{M}) ,

the total variation of μ is defined by

$$|\mu|(E) := \sup \left\{ \sum_{n=1}^{\infty} |\mu(E_n)| : \right.$$

(E_n) is a partition
of E $\left. \right\}$,

$\forall E \in \mathcal{M}$.

Remark: $|\mu|(E_1) \leq |\mu|(E_2)$ if $E_1 \subset E_2$.

Prop 5.1 If μ is a signed measure on (X, \mathcal{M}) ,
then its total variation $|\mu|$ is a finite
measure on (X, \mathcal{M}) .

Pf. First notice that $|\mu|(\emptyset) = 0$.

Next we prove that

$$|\mu|(E) = \sum_{n=1}^{\infty} |\mu|(E_n) \text{ if } (E_n) \text{ is a partition of } E.$$

Let us first prove the countable sub-additivity.

$$|\mu|(E) \leq \sum_{n=1}^{\infty} |\mu|(E_n) \quad \text{if } (E_n) \text{ is a partition of } E.$$

Let (A_k) be a partition of E .

$$\begin{aligned} \sum_{k=1}^{\infty} |\mu|(A_k) &= \sum_{k=1}^{\infty} \left| \sum_{n=1}^{\infty} \mu(A_k \cap E_n) \right| \\ &\leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} |\mu(A_k \cap E_n)| \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\mu(A_k \cap E_n)| \\ &\leq \sum_{n=1}^{\infty} |\mu|(E_n) \end{aligned}$$

Taking supremum over all partitions (A_k) of E gives

$$|\mu|(E) \leq \sum_{n=1}^{\infty} |\mu|(E_n).$$

Next we prove

$$|\mu|(E) \geq \sum_{n=1}^{\infty} |\mu|(E_n), \quad \forall \text{ a partition } (E_n) \text{ of } E$$

Clearly if $|\mu|(E_n) = \infty$ for some n , then

$$|\mu|(E) \geq |\mu|(E_n) = \infty, \text{ so the inequality holds}$$

Now we assume that $|\mu|(E_n) < \infty$ for all n .

Then for each n , \exists a partition $(E_n^k)_k$ of E_n

such that

$$|\mu|(E_n) \leq \left(\sum_{k=1}^{\infty} |\mu|(E_n^k) \right) + 2^{-n} \cdot \varepsilon$$

Hence

$$\sum_{n=1}^{\infty} |\mu|(E_n) \leq \left(\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |\mu|(E_n^k) \right) + \varepsilon$$

(Notice $(E_n^k)_{\substack{n \in \mathbb{N}, \\ 1 \leq k < \infty}}$ is a partition of E)

$$\text{Hence } \sum_{n=1}^{\infty} |\mu|(E_n) \leq |\mu|(E) + \varepsilon.$$

We obtain

$$\sum_{n=1}^{\infty} |\mu|(E_n) \leq |\mu|(E),$$

Since $\epsilon > 0$ is arbitrary.

Next we show that $|\mu|(X) < \infty$.

We need the following.

Lem 5.2. If $|\mu|(E) = \infty$ for some $E \in \mathcal{M}$,
then $\exists A, B \in \mathcal{M}$, $A \cup B = E$, A, B are disjoint,
such that

$$|\mu|(A), |\mu|(B) \geq 1 \quad \text{and} \quad |\mu|(A) = \infty.$$

We postpone the proof of Lem 5.2 until we
complete the proof of Prop 5.1.

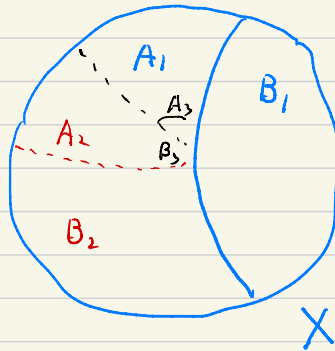
Now suppose on the contrary that

$$|\mu|(X) = \infty.$$

Then by Lem 5.2, we can find a partition
 $\{A_1, B_1\}$ of X

Such that

$$|\mu(A_1)| \geq 1, \quad |\mu(B_1)| \geq 1, \quad |\mu|(A_1) = \infty.$$



Using Lem 5.2 again, we can find a partition

$$\{A_2, B_2\} \text{ of } A_1$$

Such that

$$|\mu(A_2)|, |\mu(B_2)| \geq 1, \quad |\mu|(A_2) = \infty.$$

Continuing this process, we can find a
sequence of $\langle B_n \rangle$ such that

they are disjoint, and

$$|\mu(B_n)| \geq 1, \quad \forall n.$$

Now take $B = \bigcup_{n=1}^{\infty} B_n$. Then

$$\mu(B) = \sum_{n=1}^{\infty} \mu(B_n)$$

However, the series in the RHS diverges since

$$|\mu(B_n)| \not\rightarrow 0.$$

It leads to a contradiction. \square

Pf of Lem 5.2. Let $t > 0$.

Since $|\mu|(E) = \infty$, \exists a partition (E_n) of E such that

$$\sum_{n=1}^{\infty} |\mu(E_n)| > t.$$

\exists a large N such that

$$\sum_{n=1}^N |\mu(E_n)| > t.$$

We rearrange the sets E_n such that

$$\mu(E_1), \dots, \mu(E_m) < 0$$

and

$$\mu(E_{m+1}), \dots, \mu(E_N) \geq 0$$

Hence

$$\begin{aligned} & |\mu(E_1) + \dots + \mu(E_m)| + |\mu(E_{m+1}) + \dots + \mu(E_N)| \\ &= \sum_{n=1}^N |\mu(E_n)| > t \end{aligned}$$

WLOG, we assume that

$$|\mu(E_1) + \dots + \mu(E_m)| > \frac{t}{2}.$$

Then take $A = E_1 \cup \dots \cup E_m$

$$B = E \setminus A.$$

Clearly $|\mu(A)| \geq \frac{t}{2},$

$$|\mu(B)| = |\mu(E) - \mu(A)| \geq |\mu(A)| - |\mu(E)|$$

$$\geq \frac{t}{2} - |\mu(E)|$$

Take a large t such that

$$\frac{t}{2} - |\mu(E)| > 1.$$

Then

$$|\mu(A)|, |\mu(B)| > 1.$$

Notice that

$$\infty = |\mu|(E) = |\mu|(A) + |\mu|(B)$$

So one of $|\mu|(A), |\mu|(B)$ is ∞ .



Prop 5.3. Let (X, \mathcal{M}, μ) be a measure space.

Let $f \in L^1(\mu)$. Define

$$\lambda(E) = \int_E f \, d\mu, \quad E \in \mathcal{M}.$$

Then ① λ is a signed measure on (X, \mathcal{M}) .

② The total variation $|\lambda|$ of λ satisfies

$$|\lambda|(E) = \int_E |f| \, d\mu, \quad E \in \mathcal{M}.$$

Pf. Clearly $\lambda: \mathcal{M} \rightarrow \mathbb{R}$ is well-defined.

Let $E \in \mathcal{M}$ and (E_n) be a partition of E .

$$\text{Then } \chi_E = \lim_{n \rightarrow \infty} \sum_{k=1}^n \chi_{E_k}$$

By the Dominated Convergence Thm,

$$\begin{aligned} \int_E f \, d\mu &= \int \chi_E f \, d\mu \\ &= \lim_{n \rightarrow \infty} \int \sum_{k=1}^n \chi_{E_k} f \, d\mu \\ &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \int_{E_k} f \, d\mu \end{aligned} \quad \left. \vphantom{\int_E f \, d\mu} \right\} (*)$$

That is,

$$\begin{aligned}\lambda(E) &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \lambda(E_{nk}) \\ &= \sum_{k=1}^{\infty} \lambda(E_k).\end{aligned}$$

Hence λ is a signed measure on (X, \mathcal{M}) .

This proves (1).

Next we prove (2), i.e.

$$|\lambda|(E) = \int_E |f| d\mu, \quad E \in \mathcal{M}.$$

Let $E \in \mathcal{M}$. Let (E_n) be a partition of E

Then

$$\begin{aligned}\sum_{n=1}^{\infty} |\lambda(E_n)| &= \sum_{n=1}^{\infty} \left| \int_{E_n} f d\mu \right| \\ &\leq \sum_{n=1}^{\infty} \int_{E_n} |f| d\mu\end{aligned}$$

Taking supremum over all partitions (E_n) of E

gives

$$|\lambda|(E) \leq \sum_{n=1}^{\infty} \int_{E_n} |f| d\mu = \int_E |f| d\mu$$

(here we ^{have} used (*) in which we replace f by $|f|$)

On the other hand, let

$$A = \{x \in E : f(x) \geq 0\},$$

$$B = \{x \in E : f(x) < 0\}.$$

clearly $\{A, B\}$ is a partition of E .

$$|\lambda(A)| = \left| \int_A \underline{f} d\mu \right|$$

$$= \int_A |f| d\mu$$

$$|\lambda(B)| = \left| \int_B \underline{f} d\mu \right|$$

$$= \left| \int_B (-f) d\mu \right|$$

$$= \int_B (-f) d\mu$$

$$= \int_B |f| d\mu$$

So

$$\begin{aligned} |\lambda(A)| + |\lambda(B)| &= \int_A |f| d\mu \\ &\quad + \int_B |f| d\mu \\ &= \int (\chi_A + \chi_B) |f| d\mu \\ &= \int \chi_E |f| d\mu \\ &= \int_E |f| d\mu \end{aligned}$$

Since

$$|\lambda|(E) \geq |\lambda(A)| + |\lambda(B)| \quad \left(\begin{array}{l} \text{since } \{A, B\} \\ \text{is a partition} \\ \text{of } E \end{array} \right)$$

we obtain

$$|\lambda|(E) \geq \int_E |f| d\mu.$$

□