

Real Analysis

25-11-14

§4.4. The dual space of $L^p(\mu)$, $1 < p < \infty$.

Let $1 < p < \infty$. Take $q > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Let $g \in L^q(\mu)$. Define $\Lambda_g: L^p(\mu) \rightarrow \mathbb{R}$ by

$$(*) \quad \Lambda_g(f) = \int f g \, d\mu, \quad \forall f \in L^p(\mu).$$

By the Hölder inequality

$$\int |fg| \, d\mu \leq \|f\|_p \cdot \|g\|_q < \infty$$

Hence Λ_g is well-defined, and it is a bounded linear functional on $L^p(\mu)$.

$$\begin{aligned} \|\Lambda_g\| &= \sup_{\substack{f \in L^p(\mu) \\ \|f\|_p = 1}} |\Lambda_g(f)| \leq \sup_{\substack{f \in L^p(\mu) \\ \|f\|_p = 1}} \|g\|_q \cdot \|f\|_p \\ &= \|g\|_q. \end{aligned}$$

Define

$$L^p(\mu)' = \{ \text{bdd linear functionals on } L^p(\mu) \}$$

$$\text{Fact: } \|\Lambda_g\| = \|g\|_q, \quad \forall g \in L^q(\mu).$$

To prove the equality, we need to show $\exists f \in L^p(\mu)$
with $\|f\|_p \neq 0$ such that

$$|\Lambda_g(f)| = \|f\|_p \cdot \|g\|_q.$$

Now we take

$$f = |g|^{q-2} \cdot g = \begin{cases} 0 & \text{if } g=0 \\ |g|^{q-2} g & \text{if } g(x) \neq 0 \end{cases}$$

Notice that

$$\int |f|^p d\mu = \int |g|^{(q-1)p} d\mu = \int |g|^q d\mu < \infty$$

Hence $f \in L^p$,

$$\|f\|_p = \left(\int |g|^q d\mu \right)^{1/p} = \|g\|_q^{q/p}.$$

$$\begin{aligned}
\Lambda_g(f) &= \int g f \, d\mu \\
&= \int g \cdot |g|^{q-2} \cdot g \, d\mu = \int |g|^q \, d\mu \\
&= \|g\|_q^q = \|g\|_q \cdot \|g\|_q^{q-1} \\
&= \|g\|_q \cdot \|g\|_q^{q/p} \quad (q-1 = q/p) \\
&= \|g\|_q \cdot \|f\|_p
\end{aligned}$$

Hence $\|\Lambda_g\| \geq \|g\|_q$ and so

$$\|\Lambda_g\| = \|g\|_q.$$

Remark: Let $\Phi: L^q(\mu) \rightarrow L^p(\mu)'$ be given by

$$\Phi(g) = \Lambda_g, \quad g \in L^q(\mu)$$

Then Φ is norm preserving. So

Φ is injective. (If $\Phi(x) = \Phi(y)$,

$$\text{then } \Phi(x-y) = 0 \Rightarrow 0 = \|\Phi(x-y)\| \\ = \|x-y\|_q$$

$$\Rightarrow x = y.)$$

Thm 4.17. Let $1 < p < \infty$.

Let $\Lambda \in L^p(\mu)'$. Then \exists a unique $g \in L^q(\mu)$ such that

$$\Lambda = \Lambda_g.$$

(Hence $L^p(\mu)' \approx L^q(\mu)$)

The proof of the above theorem depends on the uniform convexity of $L^p(\mu)$.

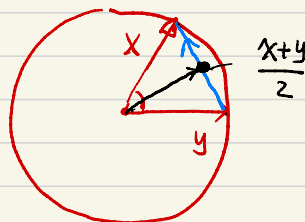
Def (Unif. Convexity).

A normed vector space X is said to be uniform convex if $\forall 0 < \varepsilon < 1$,

$\exists \theta > 0$ such that for $x, y \in X$

$$\|x\| = \|y\| = 1 \text{ and } \|x - y\| \geq \varepsilon \Rightarrow \left\| \frac{x+y}{2} \right\| < 1 - \theta.$$

Example: \mathbb{R}^2 with the standard norm is unif convex.



Example: Any Hilbert space (including \mathbb{R}^n with the standard norm)

is unif. convex.

Because
$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

(check $\langle x+y, x+y \rangle + \langle x-y, x-y \rangle$
 $= 2(\langle x, x \rangle + \langle y, y \rangle)$)

Thm 4.18. $L^p(\mu)$, $1 < p < \infty$, is unif. convex.

Prop 4.19 (Clarkson's inequality).

Let $1 < p < \infty$, $f, g \in L^p(\mu)$. Then

① If $p \geq 2$, we have

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{1}{2} (\|f\|_p^p + \|g\|_p^p)$$

② If $p \in (1, 2)$, then

$$\|f+g\|_p^q + \|f-g\|_p^q \leq 2 \left(\|f\|_p^p + \|g\|_p^p \right)^{q-1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Clarkson inequality $\Rightarrow L^p(\mu)$ is unif conv.

• $f, g \in L^p(\mu)$, $\|f\|_p = \|g\|_p = 1$.

If $p \geq 2$,

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{1}{2} \cdot 2 = 1$$

So if $\|f-g\|_p > \varepsilon$, then

$$\left\| \frac{f+g}{2} \right\|_p \leq \left(1 - \left(\frac{\varepsilon}{2} \right)^p \right)^{1/p} < 1 - \theta$$

for some $\theta > 0$.

If $p \in (1, 2)$, then

$$\|f+g\|_p^q + \|f-g\|_p^q \leq 2 \cdot (2)^{q-1} = 2^q$$

If $\|f-g\|_p > \varepsilon$, then

$$\|f+g\|_p \leq (2^q - \varepsilon^q)^{1/q}$$

$$< 2(1-\theta)$$

for some $\theta > 0$.



Proof of Clarkson's inequality in the case $p \geq 2$:

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{1}{2} (\|f\|_p^p + \|g\|_p^p),$$

$\forall f, g \in L^p(\mu)$.

It is enough to prove

$\forall x, y \in \mathbb{R},$

$$\left| \frac{x+y}{2} \right|^p + \left| \frac{x-y}{2} \right|^p \leq \frac{1}{2} (|x|^p + |y|^p)$$

Clearly the above inequality holds if $x=y=0$.

or one of them is zero

To show the general case when $x, y \neq 0$ and $|x| \geq |y|$,
of

dividing $|x|^p$ to the both sides of the inequality

and letting $z = \frac{|y|}{|x|}$, we have

$$\left(\frac{1+z}{2} \right)^p + \left(\frac{1-z}{2} \right)^p \leq \frac{1}{2} (1+z^p), \text{ for } 0 < z \leq 1$$

To prove the above inequality, let

$$g(z) = \left(\frac{1+z}{2} \right)^p + \left(\frac{1-z}{2} \right)^p - \frac{1}{2} (1+z^p).$$

$$\text{Then } g(0) = 2 \cdot \left(\frac{1}{2} \right)^p - \frac{1}{2} \leq 0$$

$$g(1) = 0$$

So to show that $g(z) \leq 0$ on $(0, 1]$,

it is enough to show that $g'(z) \geq 0$ on $(0, 1]$

Taking derivative to g gives

$$\begin{aligned} g'(z) &= p \left(\frac{1+z}{2} \right)^{p-1} \cdot \frac{1}{2} - p \left(\frac{1-z}{2} \right)^{p-1} \frac{1}{2} - \frac{1}{2} p z^{p-1} \\ &= \frac{p}{2} \left[\left(\frac{1+z}{2} \right)^{p-1} - \left(\frac{1-z}{2} \right)^{p-1} - z^{p-1} \right] \end{aligned}$$

Notice that $\frac{1+z}{2} = \frac{1-z}{2} + z$

But $(x+y)^{p-1} \geq x^{p-1} + y^{p-1}$ (since $p-1 \geq 1$)
for all $x, y \geq 0$.

(equivalent to $(1+z)^{p-1} \geq 1 + z^{p-1}$)

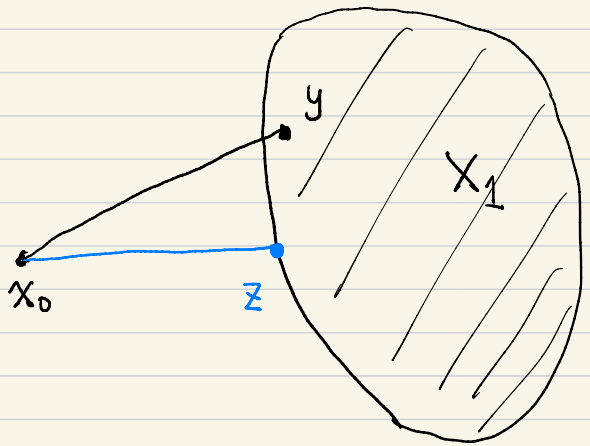
Hence $g'(z) \geq 0$ on $[0, 1]$ and we are done.

Thm 4.20. Let X be a unif. convex Banach space.

Let X_1 be a closed ^{linear} subspace of X .

Let $x_0 \in X \setminus X_1$. Then \exists a unique $z \in X_1$ such that

$$\|x_0 - z\| = \inf \{ \|x_0 - y\| : y \in X_1 \}$$



Proof: Write

$$d = \inf \{ \|x_0 - y\| : y \in X_1 \}.$$

We can choose $(y_n)_{n=1}^{\infty} \subset X_1$ such that

$$\lim_{n \rightarrow \infty} \|y_n - x_0\| = d.$$

We claim that $d \neq 0$. Otherwise $y_n \rightarrow x_0$ as $n \rightarrow \infty$.

But since $(y_n) \subset X_1$ and X_1 is closed, so $x_0 \in X_1$, which leads to a contradiction as $x_0 \notin X_2$.

Notice that

$$\limsup_{n, m \rightarrow \infty} \left\| \frac{y_n - x_0}{\|y_n - x_0\|} + \frac{y_m - x_0}{\|y_m - x_0\|} \right\| \leq 2$$

and

$$\liminf_{n, m \rightarrow \infty} \left\| \frac{y_n - x_0}{\|y_n - x_0\|} + \frac{y_m - x_0}{\|y_m - x_0\|} \right\|$$

$$= \liminf_{n, m \rightarrow \infty} \left\| \frac{y_n - x_0}{d} + \frac{y_m - x_0}{d} \right\|$$

$$= \lim_{n, m \rightarrow \infty} \frac{2}{d} \left\| \frac{y_n + y_m}{2} - x_0 \right\| \quad \left(\text{notice } \frac{y_n + y_m}{2} \in X_1 \right)$$

$$\geq \frac{2}{d} \cdot d = 2$$

Hence we obtain

$$\lim_{n, m \rightarrow \infty} \left\| \frac{y_n - x_0}{2 \|y_n - x_0\|} + \frac{y_m - x_0}{2 \|y_m - x_0\|} \right\| = 1.$$

By the unif. convexity of X , we have

$$\lim_{n, m \rightarrow \infty} \left\| \frac{y_n - x_0}{\|y_n - x_0\|} - \frac{y_m - x_0}{\|y_m - x_0\|} \right\| = 0$$

which implies

$$\lim_{n, m \rightarrow \infty} \left\| \frac{y_n - x_0}{d} - \frac{y_m - x_0}{d} \right\| = 0,$$

i.e.

$$\lim_{n, m \rightarrow \infty} \|y_n - y_m\| = 0$$

So $(y_n)_{n=1}^{\infty}$ is a Cauchy sequence in X .

Hence $\exists z \in X$ such that $y_n \rightarrow z$ as $n \rightarrow \infty$

since $(y_n) \subset X_1$ and X_1 is closed, $z \in X_1$.

Now

$$\|z - x_0\| = \lim_{n \rightarrow \infty} \|y_n - x_0\| = d.$$

Suppose \exists another $z' \in X$, so that

$$\|z' - x_0\| = d.$$

Notice that $\|z - z'\| = \varepsilon > 0$. By the unif convexity of X

$$\left\| \frac{z - x_0}{d} + \frac{z' - x_0}{d} \right\| < 1$$

$$\Rightarrow \left\| \frac{z + z'}{2} - x_0 \right\| < d$$

which is impossible, since $\frac{z + z'}{2} \in X$, so

$$\left\| \frac{z + z'}{2} - x_0 \right\| \geq d.$$

□

Lemma A: Let $\Lambda : X \rightarrow \mathbb{R}$ be a linear functional on a vector space X .

Suppose $\Lambda(x_0) \neq 0$ for some $x_0 \in X$.

Then $\forall x \in X$,

$$x - \frac{\Lambda(x)}{\Lambda(x_0)} \cdot x_0 \in \ker \Lambda,$$

$$\text{where } \ker \Lambda = \left\{ y \in X : \Lambda y = 0 \right\}.$$

$$\text{pf. } \Lambda \left(x - \frac{\Lambda(x)}{\Lambda(x_0)} x_0 \right)$$

$$= \Lambda x - \frac{\Lambda(x)}{\Lambda(x_0)} \cdot \Lambda(x_0)$$

$$= 0$$



Lemma B. Let $\Lambda, \Lambda_1 : X \rightarrow \mathbb{R}$

be linearly functionals on a vector space X . Suppose $\Lambda \neq 0$, and

$$\ker(\Lambda_1) \supset \ker \Lambda.$$

Then $\exists c \in \mathbb{R}$ such that

$$\Lambda_1 = c\Lambda.$$

Pf. Since $\Lambda \neq 0$, $\exists x_0 \in X$ such that

$$\Lambda(x_0) \neq 0.$$

By Lemma A, any vector $x \in X$

is a linear combination of x_0 and an element in $\ker \Lambda$.

$$\text{Let } c = \frac{\Lambda_1(x_0)}{\Lambda(x_0)}.$$

$$\text{set } f = \Lambda_1 - c\Lambda.$$

$$\begin{aligned}
 \text{Then } g(x_0) &= \Lambda_1(x_0) - c \Lambda(x_0) \\
 &= \Lambda_1(x_0) - \frac{\Lambda_1(x_0)}{\Lambda(x_0)} \cdot \Lambda(x_0) \\
 &= 0
 \end{aligned}$$

$$\text{and } g(y) = 0 \text{ for } y \in \ker \Lambda$$

$$\left(y \in \ker \Lambda \Rightarrow y \in \ker(\Lambda_1) \right)$$

Hence

$$\left. \begin{aligned}
 g(y) &= \Lambda_1(y) - c \Lambda(y) \\
 &= 0
 \end{aligned} \right)$$

$$\text{But } x = \frac{\Lambda(x)}{\Lambda(x_0)} \cdot x_0 + y \text{ for some } y \in \ker(\Lambda)$$

$$\begin{aligned}
 \text{Hence } g(x) &= \frac{\Lambda(x)}{\Lambda(x_0)} g(x_0) + g(y) \\
 &= 0
 \end{aligned}$$

$$\Rightarrow g = 0 \Rightarrow \Lambda_1 = c \Lambda.$$



Thm 4.17 Let $1 < p < \infty$. Let $\Lambda \in L^p(\mu)'$, then $\exists g \in L^q(\mu)$ such that $\Lambda = \Lambda g$.

Pf of Thm 4.17:

If $\Lambda \in L^p(\mu)'$ with $\Lambda = 0$.

Then we can take $g = 0$.

Next assume $\Lambda \in L^p(\mu)'$, $\Lambda \neq 0$.

Hence $\exists f_1 \in L^p(\mu)$ such that

$$\Lambda(f_1) \neq 0.$$

Notice that

$$\ker \Lambda = \{ f \in L^p(\mu) : \Lambda f = 0 \}$$

is a closed linear subspace of $L^p(\mu)$.

Since $L^p(\mu)$ is unif. convex, by Thm 4.19,

\exists a unique $h_0 \in \ker \Lambda$ such that

$$\|h_0 - f_1\|_p = \inf \{ \|f - f_1\|_p : f \in \ker(\Lambda) \}$$

This implies that for any $f \in \ker(\Lambda)$,

$$\begin{aligned}\varphi(t) &= \|h_0 + tf - f_1\|_p^p \\ &= \int |h_0 + tf - f_1|^p d\mu\end{aligned}$$

gets minimized at $t=0$.

Claim: $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable over \mathbb{R} with

$$\varphi'(t) = \int |h_0 + tf - f_1|^{p-2} (h_0 + tf - f_1) \cdot f d\mu$$

Exer. Prove the above claim.

$$\text{Hint: } \textcircled{1} \frac{d(|a+bt|^p)}{dt} = |a+bt|^{p-2} (a+bt) \cdot b$$

$$\forall a, b \in \mathbb{R}, \forall t \in \mathbb{R}.$$

$\textcircled{2}$ Use the Dominated Convergence Thm.

Now since φ is diff on \mathbb{R}

and φ takes the minimum at $t=0$,

This implies

$$\varphi'(0) = 0$$

$$\begin{aligned} \text{But } \varphi'(0) &= \int |h_0 - f_1|^{p-2} (h_0 - f_1) \cdot f \, d\mu \\ &= \int g f \, d\mu = 0. \end{aligned}$$

Next we define

$$g = |h_0 - f_1|^{p-2} \cdot (h_0 - f_1).$$

Then

$$\begin{aligned} \int |g|^q \, d\mu &= \int |h_0 - f_1|^{(p-1)q} \, d\mu \\ &= \int |h_0 - f_1|^p \, d\mu < \infty \end{aligned}$$

So $g \in L^p(\mu)$.

Since $h_0 \neq f_1$ ($\|h_0 - f_1\|_p > 0$)

we have $\|g\|_q \neq 0$.

Recall that

$$\int g f \, d\mu = 0 \quad \forall f \in \ker \Lambda.$$

Hence

$$\ker \Lambda \subset \ker \Lambda g$$

Since $\Lambda \neq 0$, by Lemma B,

$\exists c$ such that

$$\Lambda_g = c \Lambda$$

Hence $\Lambda = \frac{1}{c} \Lambda_g = \Lambda_{\frac{1}{c}g}$.

