

Real Analysis

25-11-07

Chap 4 Lebesgue spaces.

§ 4.3 Lebesgue spaces.

Let (X, M, μ) be a measure space. Let $p > 0$.

A measurable function f on X is said to be

p -integrable if

$$\int |f|^p d\mu < \infty.$$

Moreover, we write

$$\|f\|_p := \left(\int |f|^p d\mu \right)^{1/p}$$

we call it the p -norm of f .

Prop 4.10 (Hölder inequality)

Let $1 < p < \infty$. Then

$$\int |fg| d\mu \leq \left(\int |f|^p d\mu \right)^{1/p} \cdot \left(\int |g|^q d\mu \right)^{1/q},$$

where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Prop 4.11 (Minkowski inequality)

Let $p \geq 1$. Then

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

The proof of the above propositions is based on the following

(Young's inequality)

Let $\alpha, \beta \geq 0$. Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$

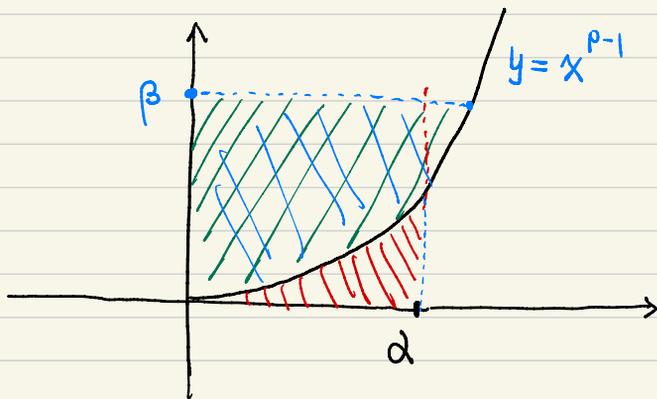
$$\text{Then } \alpha\beta \leq \frac{\alpha^p}{p} + \frac{\beta^q}{q}$$

$$"=" \text{ holds } \Leftrightarrow \beta = d^{p-1}.$$

pf of Young's inequality:

We use a geometric approach. Consider the function $y = x^{p-1}$. Its inverse is

$$x = y^{q-1} \quad (\text{using } (p-1)(q-1)=1)$$



Area of the red shaded region

$$= \int_0^d x^{p-1} dx = \frac{x^p}{p} \Big|_0^d = \frac{d^p}{p}$$

Area of the blue shaded region

$$= \int_0^{\beta} y^{q-1} dy = \frac{\beta^q}{q}$$

From the above geometry, we see that

$$a\beta \leq \frac{a^p}{p} + \frac{\beta^q}{q}$$

Clearly "=" holds $\Leftrightarrow \beta = a^{p-1}$.



Proof of the Hölder inequality:

Let $p > 1$. Let f, g be measurable functions on X . WLOG, we may assume $|f|, |g| < \infty$.

$$\text{Set } \alpha(x) = \frac{|f(x)|}{\|f\|_p}, \quad \beta(x) = \frac{|g(x)|}{\|g\|_q},$$

here $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Using Young's inequality to $d(x)$, $\beta(x)$, we obtain

$$\frac{|f(x)g(x)|}{\|f\|_p \|g\|_q} \leq \frac{|f(x)|^p}{p \cdot \|f\|_p^p} + \frac{|g(x)|^q}{q \cdot \|g\|_q^q}.$$

Taking integration w.r.t μ , we have

$$\begin{aligned} \frac{1}{\|f\|_p \|g\|_q} \int |fg| d\mu &\leq \frac{1}{p} \int \frac{|f(x)|^p}{\|f\|_p^p} d\mu(x) \\ &\quad + \frac{1}{q} \int \frac{|g(x)|^q}{\|g\|_q^q} d\mu(x) \\ &= \frac{1}{p} + \frac{1}{q} = 1, \end{aligned}$$

from which we obtain

$$\int |fg| d\mu \leq \|f\|_p \cdot \|g\|_q. \quad \square$$

Proof of the Minkowski inequality:

We prove this by applying the Hölder inequality.

If $p=1$, then since

$$|f(x) + g(x)| \leq |f(x)| + |g(x)|,$$

taking integration gives

$$\|f+g\|_1 \leq \|f\|_1 + \|g\|_1.$$

Next we assume $1 < p < \infty$.

$$|f+g|^p \leq |f| \cdot |f+g|^{p-1} + |g| \cdot |f+g|^{p-1}$$

Taking integration gives

$$\|f+g\|_p^p \leq \int |f| \cdot |f+g|^{p-1} d\mu + \int |g| \cdot |f+g|^{p-1} d\mu$$

(Using Hölder)

$$\leq \left(\int |f|^p d\mu \right)^{1/p} \cdot \left(\int |f+g|^{(p-1)q} d\mu \right)^{1/q} \\ + \left(\int |g|^p d\mu \right)^{1/p} \cdot \left(\int |f+g|^{(p-1)q} d\mu \right)^{1/q}$$

$$= \|f\|_p \cdot \|f+g\|_p^{p/q} + \|g\|_p \cdot \|f+g\|_p^{p/q} \quad (\text{using } (p-1)q = p)$$

$$= (\|f\|_p + \|g\|_p) \cdot \|f+g\|_p^{p/q}$$

Hence $\|f+g\|_p^{p - \frac{p}{q}} \leq \|f\|_p + \|g\|_p$.

Noticing that $p - \frac{p}{q} = 1$, we obtain the desired inequality. \square

Def. Let $p > 0$. Set

$$L^p(X, \mathcal{M}, \mu) = \left\{ \begin{array}{l} \text{all } p\text{-integrable functions} \\ \text{on } (X, \mathcal{M}, \mu) \end{array} \right\}$$

For short, we write $L^p(\mu) := L^p(X, \mathcal{M}, \mu)$.

Recall that for $f \in L^p(\mu)$,

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p}.$$

If $\|f\|_p = 0$, then $f = 0$ a.e.

Define $f \sim g$ if $f = g$ a.e.

Then this relation " \sim " is an equivalence relation.

Now define

$$\tilde{L}^p(\mu) = L^p(\mu) / \sim$$

For $\tilde{f} \in \tilde{L}^p(\mu)$, define

$$\|\tilde{f}\|_p = \|f\|_p \quad \text{if } \tilde{f} = [f].$$

Then $\tilde{L}^p(\mu)$ becomes a normed vector space.

Thm 4.12. Let $1 \leq p < \infty$. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L^p(\mu)$. Then $\exists f \in L^p(\mu)$ such that

$$\|f_n - f\|_p \rightarrow 0 \text{ as } n \rightarrow \infty.$$

As a consequence, $L^p(\mu)$ is a Banach space.

Pf. Since $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence, for any $j \in \mathbb{N}$, $\exists n_j \in \mathbb{N}$ such that

$$\textcircled{1} \quad \|f_n - f_m\|_p < 2^{-j} \text{ if } n, m \geq n_j.$$

We may further require that

$$n_{j+1} > n_j, \quad j=1, 2, \dots$$

By removing a subset of zero measure, we may assume $|f_n(x)| < \infty \quad \forall x \in X, n \in \mathbb{N}$.

Define

$$g_R(x) = \sum_{j=1}^R |f_{n_{j+1}}(x) - f_{n_j}(x)|.$$

$$g(x) = \sum_{j=1}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)|.$$

Clearly $g(x) = \lim_{R \rightarrow \infty} g_R(x).$

Using the Minkowski inequality to g_R gives

$$\begin{aligned} \|g_R\|_p &\leq \sum_{j=1}^R \|f_{n_{j+1}} - f_{n_j}\|_p \\ &\leq \sum_{j=1}^R 2^{-j} \quad (\text{by (1)}) \end{aligned}$$

$$< 1$$

By Fatou's lemma,

$$\begin{aligned} \|g\|_p^p &= \int |g(x)|^p d\mu(x) = \int \lim_{R \rightarrow \infty} |g_R(x)|^p d\mu(x) \\ &\leq \lim_{R \rightarrow \infty} \int |g_R(x)|^p d\mu(x) \end{aligned}$$

$$\leq 1$$

Hence $g(x) < \infty$ for μ -a.e. x .

That is, for μ -a.e. x ,

$$\sum_{j=1}^{\infty} |f_{n_{j+1}}(x) - f_{n_j}(x)| < \infty$$

Consider the sum

$$(2) \quad f_{n_1}(x) + \sum_{j=1}^{\infty} (f_{n_{j+1}}(x) - f_{n_j}(x)),$$

which converges for μ -a.e. x .

Let $f(x)$ be the above sum if (2) converges otherwise, let $f(x) = 0$.

Then for a.e $x \in X$,

$$\begin{aligned} f(x) &= \lim_{k \rightarrow \infty} \left(f_{n_k}(x) + \sum_{j=1}^k \left(f_{n_{j+1}}(x) - f_{n_j}(x) \right) \right) \\ &= \lim_{k \rightarrow \infty} f_{n_{k+1}}(x). \end{aligned}$$

That is, $f_{n_k} \rightarrow f$ a.e.

In what follows we prove that

$$\|f_n - f\|_p \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Let $\varepsilon > 0$. Take a large $N \in \mathbb{N}$ so that

$$(3) \quad \|f_n - f_m\|_p < \varepsilon \quad \forall n, m > N$$

For any $m > N$, by Fatou's lemma,

$$\begin{aligned}\|f - f_m\|_p^p &= \int |f - f_m|^p d\mu \\ &= \int \lim_{j \rightarrow \infty} |f_{n_j} - f_m|^p d\mu \\ &\leq \lim_{j \rightarrow \infty} \int |f_{n_j} - f_m|^p d\mu \\ &\leq \varepsilon^p\end{aligned}$$

That is, $\|f - f_m\|_p < \varepsilon$.

$$\begin{aligned}\text{So } \|f\|_p &\leq \|f - f_m\|_p + \|f_m\|_p \\ &< \infty\end{aligned}$$

and $\|f_n - f\|_p \rightarrow 0$ as $n \rightarrow \infty$.



Recall that a simple function on (X, \mathcal{M}, μ) is of the form

$$s = \sum_{j=1}^n \alpha_j \chi_{E_j}(x),$$

where $\alpha_j \in \mathbb{R} \setminus \{0\}$, $E_j \in \mathcal{M}$.

Let $\mathcal{S} = \left\{ s : s \text{ is a simple function} \right.$
 $\left. \sum_{j=1}^n \alpha_j \chi_{E_j} \text{ with } \mu(E_j) < \infty \right\}$.

Prop 4.14. Let $p \geq 1$. Then

\mathcal{S} is dense in $\underline{L^p(\mu)}$.

Pf. Clearly $\mathcal{S} \subset L^p(\mu)$.

Next assume $f \in L^p(\mu)$, f is non-negative.

Then \exists a sequence $(s_k)_{k=1}^{\infty}$ of non-negative simple functions, $s_k \uparrow f$.

Then by Lebesgue Dominated Convergence Thm,

$$\int |s_k - f|^p d\mu \rightarrow 0 \text{ as } k \rightarrow \infty.$$

It follows that

$$\|S_k - f\|_p \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Moreover, when k is large enough, $\|S_k\|_p < \infty$.

Writing $S_k = \sum_{j=1}^n \alpha_j \chi_{E_j}$, then

$$\alpha_j^p \mu(E_j) \leq \int |S_k|^p d\mu < \infty$$

$$\Rightarrow \mu(E_j) < \infty \Rightarrow S_k \in \mathcal{S}$$

In the general case, we write

$$f = f^+ - f^-.$$

Applying the above analysis to f^+ and f^- , we see that

$$\exists (S_k) \subset \mathcal{S} \text{ s.t. } \|S_k - f\|_p \rightarrow 0.$$

□

Prop 4.15. Let X be a LCHS, let μ be a Riesz measure. Then

$C_c(X)$ is dense in $L^p(\mu)$ for all $1 \leq p < \infty$.

Pf. Let $1 \leq p < \infty$. By Prop 4.14, it suffices to prove that for given $E \in \mathcal{M}$ with $\mu(E) < \infty$, and $\varepsilon > 0$, $\exists \varphi \in C_c(X)$ such that

$$\|\varphi - \chi_E\|_p < \varepsilon.$$

To show the above result, fix $E \in \mathcal{M}$ with $\mu(E) < \infty$, fix $\varepsilon > 0$. By Lusin's Thm, $\exists \varphi \in C_c(X)$

such that $\|\varphi\|_\infty := \sup_x |\varphi(x)| \leq 1$

and

$$\mu \{x : \varphi(x) \neq \chi_E(x)\} < 2^{-p} \varepsilon^p.$$

$$\text{Then } \|\varphi - \chi_E\|_p^p = \int |\varphi(x) - \chi_E(x)|^p d\mu$$

$$\begin{aligned}
&= \int_{\{x: \varphi(x) \neq \chi_E(x)\}} |\varphi - \chi_E|^p d\mu \\
&\leq 2^p \cdot \mu \{x: \varphi(x) \neq \chi_E(x)\} \\
&< \varepsilon^p,
\end{aligned}$$

which implies

$$\|\varphi - \chi_E\|_p < \varepsilon. \quad \square$$

Prop 4.16. $L^p(\mathbb{R}^d)$ is separable for $1 \leq p < \infty$.
 (:= $L^p(\mathcal{L}^d)$, \mathcal{L}^d - Lebesgue measure on \mathbb{R}^d)

Recall that we say a topological space X is separable if \exists a countable subset

of X which is dense in X .

pf. Let $B_n = \{x \in \mathbb{R}^d : |x| \leq n\}$, $n \in \mathbb{N}$.

Let \mathcal{P}_n denote the collection of the restriction of polynomials with rational coefficients on B_n .

That is, any element $\overset{f}{\text{of}} \mathcal{P}_n$ is of the form

$$f = \chi_{B_n} \cdot g$$

where g is a polynomial with rational coefficients defined on \mathbb{R}^d .

Hence \mathcal{P}_n is countable.

Let $\mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n$. \mathcal{P} is countable.

We show below \mathcal{P} is dense in $L^p(\mathbb{R}^d)$.

By Prop 4.15, $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.

It is enough to show that for $\varphi \in C_c(X)$,
and $\varepsilon > 0$, $\exists h \in \mathcal{P}$ s.t

$$\|\varphi - h\|_p < \varepsilon.$$

Since $\text{spt}(\varphi)$ is compact, $\exists n \in \mathbb{N}$ such
that

$$\text{spt}(\varphi) \subset B_n := \{x : \|x\| < n\}.$$

Then by Weierstrass approximation Thm,

$\exists h \in \mathcal{P}_n$ such that

$$\sup_{x \in B_n} |\varphi(x) - h(x)| < \varepsilon \cdot (\omega^d(B_n))^{-1/p}.$$

Now

$$\|\varphi - h\|_p^p = \int |\varphi(x) - h(x)|^p d\mathcal{L}^d(x)$$

$$= \int_{B_n} |\varphi(x) - h(x)|^p d\mathcal{L}^d(x)$$

$$\leq \mathcal{L}^d(B_n) \left(\sup_{x \in B_n} |\varphi(x) - h(x)| \right)^p$$

$$\leq \mathcal{L}^d(B_n) \left(\varepsilon \cdot (\mathcal{L}^d(B_n))^{-1/p} \right)^p$$

$$\leq \varepsilon^p$$

So $\|\varphi - h\|_p < \varepsilon$.

□