Real Analysis   25-10-03

Let us first recall the Riesz representation thm
            and the construction of the Riesz measure.

Thm (Riesz Representation Thm)

Let X be a LCHS. Let Λ be a positive linear
functional on $C_c(X)$. Then $\exists$ a   Borel measure
μ such that μ is finite on compact sets and

$$\Lambda(f) = \int_X f \, d\mu, \quad \forall \ f \in C_c(X).$$

Construction of μ:

For any open $G \subset X$,

$$\mu_0(G) = \begin{cases} \sup\{ \Lambda(f) : f < G \} & \text{if} \quad G \neq \phi, \\ 0 & \text{if} \quad G = \phi. \end{cases}$$

Then for each $E \subset X$, define.

$$\mu(E) = \inf\{ \mu_0(G) : G \supset E, \ G \text{ is open}\}.$$

Now we will finish the proof of the Riesz representation Thm.

Step 4. $\Lambda(f) = \int f \, d\mu$, $\forall \ f \in C_c(X)$.

It suffices to show that

$$\Lambda(f) \leq \int f \, d\mu \quad \text{for all } f \in C_c(X).$$

Because if this is true, then the reverse inequality follows by replacing f by $-f$.

Fix $f \in C_c(X)$. Suppose $f(X) \subset [a, b]$.

Let $\varepsilon > 0$.   Pick

$$y_0 < a < y_1 < \cdots < y_n = b$$

Such that

$$y_{i+1} - y_i < \varepsilon.$$

Let $K = \text{supp}(f)$.   Set

$$E_j = f^{-1}(y_{j-1}, y_j] \cap K.$$

Then

$$K = \bigcup_{j=1}^{n} E_j, \quad \text{with union being disjoint.}$$

Since K is compact, $\mu(K) < \infty$ so $\mu(E_j) < \infty$.

Choosing open G_j such that

$$\textcircled{1} \quad \mu(E_j) \geq \mu(G_j) - \frac{\varepsilon}{n}$$

② $y_{j-1} - \varepsilon < f(x) < y_j + \varepsilon$, $\forall\, x \in G_j$.

As $K \subset \bigcup_{j=1}^{n} G_j$, $\exists\ \varphi_j < G_j$ with

$$\sum_{j=1}^{n} \varphi_j = 1 \quad \text{on } K.$$

Hence
$$f = \sum_{j=1}^{n} f\,\varphi_j .$$

So
$$\Lambda(f) = \sum_{j=1}^{n} \Lambda(f\,\varphi_j)$$

$$\leq \sum_{j=1}^{n} \Lambda\big((y_j + \varepsilon)\,\varphi_j\big)$$

$$= \sum_{j=1}^{n} (y_j + \varepsilon)\ \Lambda(\varphi_j)$$

$$= \sum_{j=1}^{n} \big(|a| + y_j + \varepsilon\big)\,\Lambda(\varphi_j) - |a| \sum_{j=1}^{n} \Lambda(\varphi_j)$$

$$< \sum_{j=1}^{n} \big(|a| + y_j + \varepsilon\big)\,\mu(G_j) - |a| \sum_{j=1}^{n} \Lambda(\varphi_j)$$
$$(\text{since } |a| + y_j + \varepsilon > 0)$$

$$\leq \sum_{j=1}^{n} \big(|a| + y_{j-1} + 2\varepsilon\big)\cdot\Big(\mu(E_j) + \tfrac{\varepsilon}{n}\Big)$$
$$- |a| \sum_{j=1}^{n} \Lambda(\varphi_j)$$

$$\leq \quad \sum_{j=1}^{n} y_{j-1} \mu(E_j) \; + \; |a| \cdot \left(\sum_{j=1}^{n} \mu(E_j) - \sum_{j=1}^{n} \Lambda(\varphi_j) \right)$$

$$+ \; O(\varepsilon).$$

$$\leq \quad \int f \, d\mu \; + \; O(\varepsilon).$$

$\left(\text{since} \; \sum_{j} y_{j-1} \chi_{E_j} \leq f. \right)$

$\mu(K) \leq \sum_{j=1}^{n} \Lambda(\varphi_j)$

(Here we used $\mu(K) \leq \Lambda\left(\sum_{j} \varphi_j \right)$ by Step 3).

Regularity of Riesz measures

Def. Let μ be a Borel measure on a topological space X. A set $E \subset X$ is said to be **outer regular** if

$$\mu(E) = \inf \{ \mu(G) : G \text{ is open, } G \supset E \}$$

We say that E is **inner regular** if

$$\mu(E) = \sup \{ \mu(K) : K \text{ is compact, } K \subset E \}.$$

- Moreover, we say μ is **regular** if all measurable sets in X is both outer and inner regular with respect to μ.

Prop 2.9. Let Λ be a positive linear functional on a LCHS X. Let $\mu = \mu_\Lambda$ be the Riesz measure associated with Λ. Then the following hold:

(1) Every set in X is outer regular.

(2) Every open set in X is inner regular.

(3) Every measurable set with finite measure is inner regular.

Pf. (1) follows from the definition of μ.

Next we prove (2). Let $G \subset X$ be open. Recall that

$$\mu(G) = \sup \{ \Lambda(f) : f < G \}.$$
$$= \sup \left\{ \int_X f \, d\mu : f < G \right\}$$
$$\leq \sup \{ \mu(K) : K \text{ compact}, K \subset G \}$$

(Reason : for given $f < G$, take $K = \text{supp}(f)$.

then
$$\int_X f \, d\mu \leq \mu(K) \quad \text{since } f \leq \chi_K \,)$$

Now we prove (3), i.e every measurable set of finite measure is inner regular.

Let $A \subset X$ be measurable and $\mu(A) < \infty$.

Let $\varepsilon > 0$. Pick an open G such that $G \supset A$ and
$$\mu(G) \leq \mu(A) + \varepsilon.$$

By the additivity of μ, we obtain
$$\mu(G \backslash A) = \mu(G) - \mu(A) < \varepsilon$$

(We used the assumption
$\mu(A) < \infty$)

Pick another open $G_1 \supset G \backslash A$ such that

$$\mu(G_1) \leq \mu(G \setminus A) + \varepsilon.$$

Since $\mu(G \setminus A) = \mu(G) - \mu(A)$, the above inequality implies that

$$\mu(A) \leq \mu(G) - \mu(G_1) + \varepsilon$$

(*)

$$\leq \mu(G \setminus G_1) + \varepsilon,$$

where in the second inequality, we use

$$\mu(G \setminus G_1) + \mu(G_1) \geq \mu(G).$$

Since $A \subset G$, it follows that

$$A = G \setminus (G \setminus A)$$

$$\supset G \setminus G_1 \qquad (\text{since } G_1 \supset G \setminus A).$$

Now take a compact set $K \subset G$ such that

$$\mu(G \setminus K) < \varepsilon.$$

Take $\widehat{K} = K \backslash G_1$. Then $\widehat{K}$ is compact, and

$$A \supseteq G \backslash G_1 \supseteq \widehat{K}.$$

Using the fact that $(K \backslash G_1) \cup (G \backslash K) \supset G \backslash G_1$
We obtain

$$\mu(\widehat{K}) = \mu(K \backslash G_1) \geqslant \mu(G \backslash G_1) - \mu(G \backslash K)$$

$$\geqslant \mu(A) - \varepsilon - \varepsilon$$

(by (✳) and $\mu(G \backslash K) < \varepsilon$)

Hence A is inner regular. This proves (3).

∎

Prop 2.10. Let μ be a Riesz measure on a LCHS X which is σ-finite with respect to μ.

$$\left(\text{i.e.} \quad X = \bigcup_j X_j \quad \text{with} \quad \mu(X_j) < \infty\right)$$

Then the following hold:

(1) For any measurable $E \subset X$ and $\varepsilon > 0$, there exist an open set G and a closed set F so that

$$F \subset E \subset G, \quad \text{and} \quad \mu(G \setminus F) < \varepsilon.$$

(2). For any measurable set $E \subset X$, there exists a G_δ set A and a F_σ set B such that $B \subset E \subset A$ and $\mu(A \setminus B) = 0.$

Consequently, $\mathcal{M}_C$ is the completion of $\mathcal{B}_X$.

(3) Every measurable set is inner regular.

Recall that a G_δ set is a countable intersection of open sets; a F_σ set is a union of countably many closed sets).

pf. Let $E \subset X$ be measurable.

Let $X = \bigcup_j X_j$ with $\mu(X_j) < \infty$.

Write $E_j = X_j \cap E$. Then $\mu(E_j) < \infty$. Now let $\varepsilon > 0$.

Pick open set $G_j \supset E_j$ so that
$$\mu(G_j \setminus E_j) < \varepsilon \, 2^{-j}.$$

Let $G = \bigcup\limits_{j=1}^{\infty} G_j$.

Then $\quad G \backslash E = \left(\bigcup\limits_{j} G_j \right) \backslash E$

$$\subset \bigcup\limits_{j} \left(G_j \backslash E_j \right)$$

Hence

$$\mu(G \backslash E) \leq \sum\limits_{j} \mu(G_j \backslash E_j) \; < \; \sum\limits_{j} \cdot \varepsilon \, 2^{-j} = \varepsilon.$$

Using a similar argument for E^c,

we can find an open $G_1 \supset E^c$

such that

$$\mu\left(G_1 \backslash (E^c) \right) < \varepsilon.$$

Notice that $\quad G_1 \backslash (E^c) = G_1 \cap E$

$$= E \backslash G_1^c$$

Set $F = G_1^c$.   Then F is closed

and
$$E \backslash F = G_1 \backslash E^c$$

Hence $\mu(E \backslash F) < \varepsilon$.

Since $F \subset E \subset G$,

we have
$$\mu(G \backslash F) = \mu(G \backslash E) + \mu(E \backslash F)$$
$$< 2\varepsilon.$$
$$(\text{because } G \backslash F = (G \backslash E) \cup (E \backslash F))$$

This proves (1).

Next we prove (2).   By (1), we can find
open sets (G_n), closed sets (F_n) such that
$$\begin{cases} F_n \subset E \subset G_n, \quad n \in \mathbb{N} \\ \mu(G_n \backslash F_n) < 2^{-n} \end{cases}$$

Let $\quad A = \bigcap\limits_n G_n$,

$\qquad B = \bigcup\limits_n F_n$,

Then $\;A\;$ is a $\;G_\delta\;$ set and $\;B\;$ is a $\;F_\sigma\;$ set.

Clearly $\qquad B \subset E \subset A$, and

$$\mu\left(A \backslash B \right) \le \mu\left(G_n \backslash F_n \right) < 2^{-n},$$

which implies $\;\mu(A \backslash B) = 0$.

This proves (2).

Finally we prove (3). It suffices to prove

$$\sup\left\{ \mu(K) : \; K \text{ compact}, \; K \subset E \right\} = \infty$$

if $\;\mu(E) = +\infty$.

For this, let $\;X = \bigcup\limits_j X_j\;$ with $\;\mu(X_j) < \infty$

Letting $E_j = E \cap X_j$, we have

$$E = \bigcup_{j=1}^{\infty} E_j$$

Hence $\mu\left(\bigcup_{j=1}^{\infty} E_j \right) = \infty$, which implies

$$\mu\left(\bigcup_{i=1}^{N} E_j \right) \to \infty \quad \text{as } N \to \infty.$$

Now for each N, $\boxed{\mu\left(\bigcup_{i=1}^{N} E_j \right) < \infty}$ we can find a compact

$$K_N \subset \bigcup_{j=1}^{N} E_j \quad \text{with}$$

$$\mu(K_N) \geqslant \mu\left(\bigcup_{j=1}^{N} E_j \right) - \frac{1}{N}.$$

$$\to +\infty \quad \text{as } N \to \infty.$$

$\boxed{\square}$

§ 2.5.   Lusin's Thm.

Thm 2.12.   Let μ be a Riesz measure on a LCHS X.   Let $f: X \to \mathbb{R}$ be measurable such that f vanishes on A^c for some measurable set A with finite measure. such that   Then for any $\varepsilon > 0$, $\exists$ $g \in C_c(X)$,

$$\mu \left\{ x: \quad f(x) \neq g(x) \right\} < \varepsilon.$$

Pf.   Writing $f = f^+ - f^-$, we may simply assume f is non-negative.

Also we may assume f is bounded and A is compact by an approximation argument.

Dividing f by a large number, we may assume $0 \leqslant f < 1$.

Next we construct simple functions $S_n \uparrow f$.

Such that for $n = 1, 2, \cdots,$

$$S_n(x) = \frac{j}{2^n}, \qquad \text{if} \quad \frac{j}{2^n} \leq f(x) < \frac{j+1}{2^n}$$

for some $j = 0, 1, \cdots, 2^n - 1$.

Notice that $S_1(x) = \frac{1}{2}$ or 0 and

$$S_n(x) - S_{n-1}(x) = \frac{1}{2^n} \quad \text{or} \quad 0 \text{ for } n \geq 2.$$

Letting $S_0(x) = 0$, then

$$S_n(x) - S_{n-1}(x) = \frac{1}{2^n} \chi_{T_n}(x), \quad n \geq 1,$$

where T_n is the set of points x at which

$$S_n(x) - S_{n-1}(x) = \frac{1}{2^n}.$$

Notice that $T_n \subset A$.

(because on A^c, $f(x) = 0$ so $S_n(x) = 0$)

Next notice that

$$f(x) = \sum_{n=1}^{\infty} \left(S_n(x) - S_{n-1}(x) \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \chi_{T_n}(x).$$

Since A is compact, we can choose an open set V such that

$$A \subset V, \quad \overline{V} \text{ is compact.}$$

Now let $\varepsilon > 0$. For each n, pick an open set G_n and compact K_n such that

$$K_n \subset T_n \subset G_n \subset V \subset \overline{V},$$

So that $\mu(G_n \setminus K_n) < \dfrac{\varepsilon}{2^n}$.

By the Urysohn lemma, $\exists$ $h_n \in C_c(X)$

$$K_n < h_n < G_n$$

Now we define

$$g = \sum_{n=1}^{\infty} \frac{1}{2^n} h_n.$$

Hence $g \in C_c(X)$. (Because $g = 0$ on $\overline{V}^c$)

Notice that

$$h_n = \chi_{T_n} \quad \text{except on the set} \quad G_n \backslash K_n.$$

(Since on K_n, $h_n = \chi_{T_n} = 1$
on G_n^c, $h_n = 0 = \chi_{T_n}$)

Hence $f = g$ except on $\bigcup_n \left(G_n \backslash K_n \right)$

It follows that

$$\{x: f_{(x)} \neq g_{(x)}\} \subset \bigcup_n G_n \backslash K_n$$

However,

$$\mu\left(\bigcup_n G_n \backslash K_n\right) \leq \sum_n \mu\left(G_n \backslash K_n\right)$$

$$< \sum_n \varepsilon \cdot 2^{-n} = \varepsilon.$$

Corollary 2.13.

Under the assumption of Thm 2.12,

let f be a measurable function satisfying

$|f| \leq 1$.

then $\exists$ a sequence $(g_n) \subset C_c(X)$

such that

$$\lim_{n \to \infty} g_n(x) = f_{(x)} \quad a.e.$$

Pf. By Lusin's Thm, we can find for $n \in \mathbb{N}$,

$$g_n \in C_c(X) \quad \text{and} \quad E_n \subset X \text{ measurable}$$

such that

$$\mu(E_n) < \frac{1}{2^n} \quad \text{and} \quad f = g_n \text{ on } E_n^c.$$

Then

$$\sum_{n=1}^{\infty} \mu(E_n) < \infty.$$

By Borel-Cantelli lemma,

$$\mu\left\{ x: \ x \in E_n \text{ for infinitely many } n \right\} = 0$$

Hence for almost all point x,

x belongs to finitely many E_n's

and let $n_0(x)$ be the largest such n.

Then
$$g_n(x) = f(x) \quad \text{for all } n \geq n_0(x).$$

Hence $\quad \lim_{n \to \infty} g_n(x) = f(x).$ ▨