Real Analysis 25-09-26

Review

- Outer measure ($\mu(\emptyset)=0$, $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$ if $A \subseteq \bigcup_{i=1}^{\infty} A_i$)

- (Caratheodory's Thm) Let μ be an outer measure on X.

Then $(X, \mathcal{M}_c, \mu)$ is a complete measure space,
where $\mathcal{M}_c = \{ E \subset X : E \text{ is } \mu\text{-measurable} \}$.

§ 2.2 Topological and metric spaces.

- Topological spaces
- Metric spaces.

Prop. 2.3 Let $(X, \mathcal{M}, \mu)$ be a measure space, where X is a topological space and assume that

$$\mathcal{M} \supseteq \mathcal{B}_X \quad (\text{where } \mathcal{B}_X \text{ is the Borel } \sigma\text{-algebra on } X)$$

Then any continuous function $f: X \to \mathbb{R}$ (or $\overline{\mathbb{R}}$) is $\mathcal{M}$-measurable.

Pf. $\forall$ open $G \subset \mathbb{R}$ (or $\overline{\mathbb{R}}$), by continuity,

$f^{-1}(G)$ is open in X

hence $f^{-1}(G) \in \mathcal{B}_X \subset \mathcal{M}$. $\square$.

Def. (Borel measure)

An outer measure μ on a topological space X is said to be a Borel measure if all Borel sets are μ-measurable.

Prop 2.4 (Caratheodory's criterion)

Let (X, d) be a metric space. Let μ be an outer measure on X. Suppose that μ satisfies

$$(*) \quad \mu(A \cup B) = \mu(A) + \mu(B) \text{ if } d(A, B) > 0,$$

where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

(An outer measure μ satisfying $(*)$ is called a metric outer measure)

Then μ is a Borel measure.

Proof. It suffices to show that all closed sets in X are μ-measurable.

Let A be a closed set. We need to show

$$\mu(C) \geq \mu(C \cap A) + \mu(C \setminus A), \quad \forall\, C \subset X.$$

For $n \in \mathbb{N}$, define

$$A_n = \{ x \in X : \ d(x, A) \leq \tfrac{1}{n} \},$$

where $d(x, A) := \inf \{ d(x, y) : y \in A \}$.

Then A_n are closed, $A_n \searrow A$.

Notice that $d(A, A_n^c) \geq \frac{1}{n}$. So

$$d(C \cap A, C \setminus A_n) \geq \frac{1}{n}.$$

Hence
$$\begin{aligned} \mu(C) &\geq \mu\big((C \cap A) \cup (C \setminus A_n) \big) \\ &= \mu(C \cap A) + \mu(C \setminus A_n), \quad \forall\, n. \end{aligned}$$

Next we show that

$$(**) \qquad \varliminf_{n\to\infty} \mu(C\setminus A_n) \geq \mu(C\setminus A),$$

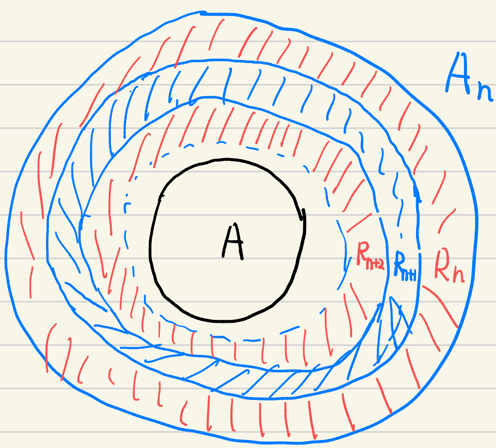
which implies $\quad \mu(C) \geq \mu(C\cap A) + \mu(C\setminus A)$.

To prove $(**)$, define for $k\in\mathbb{N}$,

$$R_k = \left\{ x\in X : \frac{1}{1+k} < d(x,A) \leq \frac{1}{k} \right\}.$$

Then

$$A_n = A \cup \left(\bigcup_{k=n}^{\infty} R_n \right) \quad \text{with union being disjoint.}$$



Then $\quad A^c = A_n^c \cup (A_n\setminus A)$

So

$$C\setminus A \subset (C\setminus A_n) \cup (C\cap (A_n\setminus A))$$

$$= (C\setminus A_n) \cup \left(C\cap \left(\bigcup_{k=n}^{\infty} R_k\right)\right).$$

Hence

$$\mu(C\setminus A) \leq \mu(C\setminus A_n) + \mu\left(C\cap \left(\bigcup_{k=n}^{\infty} R_k\right)\right).$$

To show that $\varliminf_{n\to\infty} \mu(C\setminus A_n) \geq \mu(C\setminus A)$,

it is enough to show

$$\sum_{k=1}^{\infty} \mu(C\cap R_k) < \infty$$

(which implies $\mu(C\cap \bigcup_{k=n}^{\infty} R_k) \to 0$ as $n\to\infty$.)

Notice that $R_2, R_4, \cdots, R_{2k}, \cdots$, have positive distance between them, So are $R_1, R_3, R_5, \dots$

Hence

$$
\begin{aligned}
&\mu(C) \\
&\geq \mu\left((C \cap R_2) \cup (C \cap R_4) \cup \cdots (C \cap R_{2k})\right) \\
&= \mu(C \cap R_2) + \mu\left((C \cap R_4) \cup \cdots \cup (C \cap R_{2k})\right) \\
&= \cdots \\
&= \mu(C \cap R_2) + \cdots + \mu(C \cap R_{2k}).
\end{aligned}
$$

Hence

$$\sum_{k=1}^{\infty} \mu(C \cap R_{2k}) \leq \mu(C) < \infty.$$

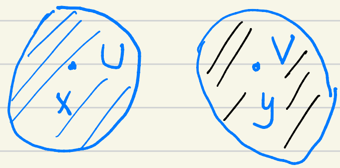
Similarly

$$\sum_{k=1}^{\infty} \mu(C \cap R_{2k-1}) \leq \mu(C) < \infty$$

Therefore $\displaystyle\sum_{k=1}^{\infty} \mu(C \cap R_k) \leq 2 \cdot \mu(C) < \infty$ ◻

§ 2.3 Locally compact Hausdorff spaces.

Def. A topological space X is said to be a Hausdorff space if $\forall\ x, y \in X$ with $x \neq y$,
$\exists$ open sets U and V such that
$x \in U$, $y \in V$ and $U \cap V = \emptyset$.



Def. A topological space is said to be locally compact if $\forall\ x \in X$, $\exists$ an open U such that
$x \in U$ and $\overline{U}$ is compact.
(closure of U).

Notation: X is said to be a LCHS if X is a Hausdorff space and locally compact.

Remark:
- $\mathbb{R}^n$ is a LCHS.
- All compact metric spaces are LCHS.

Prop 2.5. Let X be a LCHS.

Let $K \subset G$ where K is compact, G is open in X.

Then $\exists$ open V such that

$$K \subset V \subset \overline{V} \subset G.$$

Thm 2.6 (Urysohn's lemma).

Let X be a LCHS.

Let $K \subset G$, K compact, G open.

Then there exists a cts $f: X \to \mathbb{R}$ such that

- $\text{supp}(f) \subset G$
- $0 \le f \le 1$ on X
- $f(x) = 1$ for all $x \in K$

where

$$\text{supp}(f) = \overline{\{x : f(x) \neq 0\}}.$$

Thm 2.7 (partition of Unity).

Let X be a LCHS.

Suppose $K \subset \bigcup_{k=1}^{N} G_k$, with G_k open

where K is compact.

There exist $\{\varphi_j\}_{j=1}^{N} \subset C(X)$,

such that

$$\varphi_j < G_j \quad \text{and} \quad \sum_{j=1}^{N} \varphi_j = 1 \text{ on } K$$

where $\varphi_j < G_j$ means that

① $\operatorname{supp}(\varphi_j) \subset G_j$; and

② $0 \leq \varphi_j \leq 1$ on X.

§ 2.4 Riesz representation Thm.

- For a topologial space X, let

$$C_c(X) = \{ f \in C(X) : \operatorname{supp}(f) \text{ is compact} \}.$$

clearly, $C_c(X)$ is a vector space.

(that is, $af + bg \in C_c(X)$ if $f, g \in C_c(X)$ and $a, b \in \mathbb{R}$)

- A linear functional on a vector space is simply a linear map from the vector space to $\mathbb{R}$.

- A linear functional Λ on $C_c(X)$ is called positive if

$$\Lambda(f) \geq 0 \quad \text{if} \quad f \geq 0.$$

Example: Let X be a topological space. Let μ be a Borel measure on X such that $\mu(K) < \infty$ for all compact sets K. Define

$$\Lambda(f) = \int_X f \, d\mu, \quad \forall f \in C_c(X).$$

Then Λ is a positive linear functional on $C_c(X)$.

Justification: It is easy to show the positivity and linearity of Λ.

Below we show that $\Lambda(f) \in \mathbb{R}$ for $f \in C_c(X)$.

Let $f \in C_c(X)$ and $K = \mathrm{supp}(f)$.

Then K is compact.

Hence

$$\sup_{x \in X} |f(x)| = \sup_{x \in K} |f(x)| < \infty.$$

It follows that

$$\int_X f \, d\mu = \int_K f \, d\mu.$$

and

$$\begin{aligned} \left| \int_K f \, d\mu \right| &\leq \int_K |f| \, d\mu \\ &\leq \mu(K) \cdot \sup_{x \in K} |f(x)| \\ &< \infty. \end{aligned}$$

Thm 2.8 (Riesz representation Thm)

Let X be a LCHS. Let Λ be a positive linear functional on $C_c(X)$. Then $\exists$ a Borel measure μ on X such that μ is finite on every compact set, and

$$\Lambda(f) = \int f \, d\mu, \qquad \forall \, f \in C_c(X).$$

- Before the proof, we construct a measure μ from Λ.

Let $G \subset X$ be non-empty and open. We define

$$\mu_0(G) = \sup\{ \Lambda(f) : f \prec G \}.$$

(Recall $f \prec G$ means that $f \in C_c(X)$, $0 \leq f \leq 1$, $\operatorname{supp}(f) \subset G$. By Urysohn Lem, $\exists$ $f \in C_c(X)$ such that $f \prec G$)

Next set $\mu_0(\emptyset) = 0$.

Now for any $E \subset X$, define

$$\mu(E) = \inf\{\mu_0(G) : G \text{ is open, } G \supset E\}.$$

Proof of Riesz representation Thm (Thm 2.8):

First observe that

$$\mu_0(G_1) \leq \mu_0(G_2) \text{ for open set } G_1, G_2 \text{ with } G_1 \subset G_2.$$

As a direct consequence, we have

(i) $\mu(G) = \mu_0(G)$ for open $G \subset X$.

(ii) $\mu(E_1) \leq \mu(E_2)$ if $E_1 \subset E_2$.

Next we prove the theorem in 4 steps.

① μ is an outer measure.

② all Borel sets are μ-measurable.

③ $\mu(K) < \infty$ for compact K.

④ $\Lambda(f) = \int f \, d\mu$, $f \in C_c(X)$.

Step 1. μ is an outer measure.

We need to show that

$$\mu(E) \leq \sum_{j=1}^{\infty} \mu(E_j) \text{ if } E \subset \bigcup_{j=1}^{\infty} E_j.$$

We may assume $\sum_{j=1}^{\infty} \mu(E_j) < \infty$.

Let $\varepsilon > 0$. Pick open $G_j \supset E_j$ such that

$$\mu(E_j) > \mu_0(G_j) - \frac{\varepsilon}{2^j}, \quad j = 1, 2, \ldots$$

Set $G = \bigcup_{j=1}^{\infty} G_j$. Then G is open.

Now we estimate $\mu_0(G)$. Let $f \prec G$.

Let $K = \text{supp}(f)$. Then K is compact.

Since $K \subset G = \bigcup_{j=1}^{\infty} G_j$, by the compactness of K,

$\exists\ N$ such that

$$K \subset \bigcup_{j=1}^{N} G_j .$$

Then by the theorem of partition of unity,

$\exists\ \varphi_j \prec G_j$ such that

$$\sum_{j=1}^{N} \varphi_j = 1 \quad \text{on } K.$$

We obtain that

$$f = \sum_{j=1}^{N} f \cdot \varphi_j \quad \text{on } X.$$

Hence

$$\begin{aligned} \Lambda(f) &= \sum_{j=1}^{N} \Lambda(f\varphi_j) \\ &\leq \sum_{j=1}^{N} \mu_0(G_j) \quad (\text{since } f\varphi_j \prec G_j) \\ &\leq \sum_{j=1}^{\infty} \mu_0(G_j). \end{aligned}$$

Since f is arbitrarily taken with $f \prec G$, we obtain

$$\mu_0(G) \leq \sum_{j=1}^{\infty} \mu_0(G_j)$$

Hence

$$\mu(E) \leq \mu_0(G) \leq \sum_{j=1}^{\infty} \mu_0(G_j)$$

$$\leq \sum_{j=1}^{\infty} \left(\mu(E_j) + \frac{\varepsilon}{2^j} \right)$$

$$= \left(\sum_{j=1}^{\infty} \mu(E_j) \right) + \varepsilon$$

Letting $\varepsilon \to 0$ gives

$$\mu(E) \leq \sum_{j=1}^{\infty} \mu(E_j).$$

Step 2. μ is a Borel measure.

Equivalently, we need to show that all open sets are μ-measurable.

Let $U \subset X$ be open. We need to prove

$$\mu(C) \geq \mu(C \cap U) + \mu(C \setminus U), \ \forall C \subset X.$$

By the definition of μ, it is enough to prove

$$(*) \quad \mu(G) \geq \mu(G \cap U) + \mu(G \setminus U), \ \forall \text{ open } G$$

(because if this is true, then $\forall \varepsilon > 0$, pick open $G \supset C$ such that $\mu(C) \geq \mu(G) - \varepsilon$. Then by $(*)$,

$$\mu(C) \geq \mu(G) - \varepsilon \geq \mu(G \cap U) + \mu(G \setminus U) - \varepsilon$$
$$\geq \mu(C \cap U) + \mu(C \setminus U) - \varepsilon.)$$

To prove $(*)$, we may assume $\mu(G) < \infty$.
Let $\varepsilon > 0$, and pick $\varphi \prec G \cap U$ such that

$$\Lambda(\varphi) \geq \mu_0(G \cap U) - \varepsilon.$$

Let $K = \operatorname{supp}(\varphi)$. Pick $\psi \prec G \setminus K$.

Since $\operatorname{supp}(\psi)$ and K are disjoint,

$$\varphi + \psi \prec G.$$

Hence

$$\begin{aligned}
\mu(G) = \mu_0(G) &\geq \Lambda(\varphi + \psi) \\
&= \Lambda(\varphi) + \Lambda(\psi) \\
&\geq \mu(G \cap U) - \varepsilon + \Lambda(\psi).
\end{aligned}$$

Recall that $\psi \prec G \setminus K$ is arbitrily taken,
we obtain

$$\mu(G) \geq \mu(G \cap U) - \varepsilon + \mu(G \setminus K)$$

$$\geq \mu(G \cap U) - \varepsilon + \mu(G \setminus U)$$

(since $G \setminus K \supset G \setminus U$).

Letting $\varepsilon \to 0$ gives

$$\mu(G) \geq \mu(G \cap U) + \mu(G \setminus U).$$

Step 3. μ is finite on compact sets.

We shall prove

$$\mu(K) = \inf \{ \Lambda(f) : K \prec f \}$$

for all compact sets K,

where $K \prec f$ means $f \in C_c(X)$,

$0 \leq f \leq 1$ on X and $f = 1$ on K.

We first show $\mu(K) \leq \inf\{\Lambda(f) : K \prec f\}$.

Let $f \in C_c(X)$ such that $K \prec f$.

For $\alpha \in (0,1)$, define

$$G_\alpha = \{x \in X : f(x) > \alpha\}.$$

Then G_α is open and $G_\alpha \supset K$.

Let $\varphi \prec G_\alpha$. Then

$$\varphi < \frac{f}{\alpha} \quad \text{on } G_\alpha$$

Hence

$$\varphi \leq \frac{f}{\alpha} \quad \text{on } X.$$

It follows that

$$\Lambda(\varphi) \leq \Lambda\left(\frac{f}{\alpha}\right) = \frac{1}{\alpha}\Lambda(f)$$

(here we used the positivity of Λ)

Hence

$$\mu(G_\alpha) \leq \frac{1}{\alpha}\Lambda(f).$$

In particular

$$\mu(K) \leq \mu(G_\alpha) \leq \frac{1}{\alpha} \Lambda(f).$$

Letting $\alpha \uparrow 1$ gives $\mu(K) \leq \Lambda(f)$.

This showes $\mu(K) \leq \inf\{\Lambda(f) : K < f\}$.

To show the other direction, $\forall \varepsilon > 0$,
we can find open $G \supset K$ such that

$$\mu(K) \geq \mu_0(G) - \varepsilon = \mu(G) - \varepsilon.$$

By Urysohn's lemma, $\exists\ f \in C_c(X)$ such that

$$K < f < G.$$

Hence

$$\begin{aligned} \mu(K) &\geq \mu(G) - \varepsilon \\ &\geq \Lambda(f) - \varepsilon. \\ &\geq \inf\{\Lambda(g) : K < g\} - \varepsilon \end{aligned}$$

Letting $\varepsilon \to 0$ gives the desired inequality.