

Real Analysis

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§1.4 Integration on measure spaces.

Let (X, \mathcal{M}, μ) be a measure space.

Let s be a simple function, i.e.

$$s = \sum_{i=1}^N \alpha_i \chi_{A_i},$$

with $\alpha_1 < \alpha_2 < \dots < \alpha_N$, $A_i = \{x \in X : s(x) = \alpha_i\} \in \mathcal{M}$.

Def. Let $s = \sum_{i=1}^N \alpha_i \chi_{A_i}$ (in its standard form) be a non-negative simple function. Then we define

$$\int_E s \, d\mu = \sum_{i=1}^N \alpha_i \mu(A_i \cap E), \quad \forall E \in \mathcal{M}.$$

Prop 1.7. Let $s = \sum_{i=1}^N \gamma_i \chi_{E_i}$ be a non-negative simple function (in a general form). Then

$$(*) \quad \int_E s \, d\mu = \sum_{i=1}^N \gamma_i \mu(E_i \cap E), \quad \forall E \in \mathcal{M}.$$

Consequently,

$$\int_E s + t \, d\mu = \int_E s \, d\mu + \int_E t \, d\mu$$

for any other non-negative simple function t .

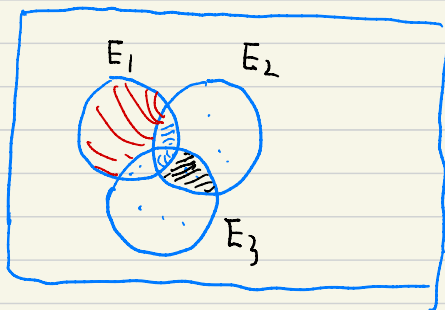
Pf. First observe that (*) holds if E_i are disjoint.

Next we prove (*) in the general case that E_i may not be disjoint. The main idea is to construct (F_j) , ^{so that} F_j are disjoint, and each E_i is the union of those F_j containing i in E_i

$$\text{(i.e. } E_i = \bigcup_{j: F_j \subset E_i} F_j \text{)}$$

Indeed each F_j can be written as

$$A_1 \cap A_2 \cap \dots \cap A_N : \quad A_i = E_i \text{ or } E_i^c$$



Now

$$\begin{aligned} S &= \sum_i \chi_i \chi_{E_i} \\ &= \sum_i \chi_i \left(\sum_{j: F_j \subset E_i} \chi_{F_j} \right) \end{aligned}$$

$$= \sum_j \left(\sum_{i: E_i \supset F_j} \gamma_i \right) \cdot \chi_{F_j}$$

$$= \sum_j \beta_j \chi_{F_j} \quad (\text{where } \beta_j = \sum_{i: E_i \supset F_j} \gamma_i)$$

$$\text{Hence } \int_E s \, d\mu = \sum_j \beta_j \mu(F_j \cap E)$$

$$= \sum_j \left(\sum_{i: E_i \supset F_j} \gamma_i \right) \mu(F_j \cap E)$$

$$= \sum_i \left(\sum_{j: F_j \subset E_i} \mu(F_j \cap E) \right) \gamma_i$$

$$= \sum_i \mu(E_i \cap E) \cdot \gamma_i.$$

□

Next we define the integration for non-negative measurable functions.

Def: Let $f: X \rightarrow [0, \infty]$ be measurable.

We define for $E \in \mathcal{M}$,

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu : 0 \leq s \leq f, s \text{ is simple} \right\}.$$

Remark: Alternatively, we can define

$$\int_E f d\mu = \sup \left\{ \int_E s d\mu : s \leq f \text{ a.e., } s \text{ is non-negative simple} \right\}.$$

where $s \leq f$ a.e. means $\exists N \in \mathcal{M}$ with $\mu(N) = 0$
so that $s \leq f$ on $X \setminus N$.

(here we use the fact if $s \leq f$ a.e.,
then taking $\tilde{s} = s \cdot \chi_{X \setminus N}$, then $\tilde{s} \leq f$
and $\int_E \tilde{s} d\mu = \int_E s d\mu$).

Prop 1.8. Let $f, g : X \rightarrow [0, +\infty]$ measurable. Then

$$(1) \int_E f d\mu = \int_X f \cdot \chi_E d\mu, \quad \forall E \in \mathcal{M}$$

$$(2) \int_X g d\mu \geq \int_X f d\mu \text{ if } g \geq f \text{ a.e.}$$

Moreover, if $\int_X g d\mu < \infty$, then " \geq " holds
iff $g = f$ a.e.

$$(3) \int_{E_1} f \, d\mu \leq \int_{E_2} f \, d\mu \quad \text{if } E_1 \subset E_2$$

$$(4) c \int_X f \, d\mu = \int_X cf \, d\mu \quad \forall c \geq 0.$$

Pf. Here we only prove (1), i.e

$$\int_E f \, d\mu = \int_X f \chi_E \, d\mu. \quad (**).$$

We first prove that (**) holds if f is simple.

To see it, let

$$s = \sum_i d_i \chi_{A_i}$$

Then

$$\begin{aligned} \int_E s \, d\mu &= \sum_i d_i \mu(A_i \cap E) \\ &= \int_X s \cdot \chi_E \, d\mu. \end{aligned}$$

Next we consider the general case. We prove

that

$$\int_E f \, d\mu \geq \int_X f \chi_E \, d\mu.$$

To see this, let $0 \leq s = \sum \alpha_i \chi_{A_i} \leq f \chi_E$

Then $s \cdot \chi_E = s$ and $s \leq f$.

(since $s(x) = 0$ if $x \notin E$)

$$\text{Hence } \int_E f \, d\mu \geq \int_E s \, d\mu$$

$$= \int_X s \chi_E \, d\mu$$

$$= \int_X s \, d\mu,$$

taking supremum of $\int_X s \, d\mu$ over $0 \leq s \leq f \chi_E$

gives

$$\int_E f \, d\mu \geq \int_X f \chi_E \, d\mu.$$

Next we prove $\int_E f \, d\mu \leq \int_X f \chi_E \, d\mu$.

To see it, let $0 \leq s \leq f$, where s is simple.

Then $s \chi_E \leq f \chi_E$, so

$$\int_X f \chi_E \, d\mu \geq \int_X s \chi_E \, d\mu = \int_E s \, d\mu,$$

taking supremum over $0 \leq s \leq f$ gives

$$\int_X f \chi_E d\mu \geq \int_E f d\mu. \quad \square$$

Prop 1.9 (Markov inequality)

Let $f: X \rightarrow [0, +\infty]$ measurable.

Let $M > 0$. Then

$$\mu\{x: f(x) \geq M\} \leq \frac{1}{M} \int_X f d\mu.$$

Consequently (i) If $\int_X f d\mu < \infty$, then
 f is finite a.e.

(ii) If $\int_X f d\mu = 0$, then
 $f = 0$ a.e.

Pf. Write $E_M := \{x: f(x) \geq M\}$.

Then $f \geq M \cdot \chi_{E_M}$

Taking integration gives

$$\int_X f \, d\mu \geq \int_X M \chi_{E_M} \, d\mu = M \mu(E_M).$$

Hence
$$\mu(E_M) \leq \frac{1}{M} \int_X f \, d\mu.$$

Next assume $\int_X f \, d\mu < \infty$.

Write

$$E_\infty = \{x : f(x) = +\infty\}. \text{ Then}$$

$$E_\infty \subset E_M \quad \forall M > 0$$

So

$$\mu(E_\infty) \leq \mu(E_M) \leq \frac{1}{M} \int_X f \, d\mu$$

Letting $M \rightarrow +\infty$ gives $\mu(E_\infty) = 0$, i.e.

f is finite a.e.

Finally assume $\int f \, d\mu = 0$.

$$\text{Let } A = \{x : f(x) > 0\}.$$

$$\text{Then } A = \bigcup_{n=1}^{\infty} E_{1/n}$$

(clearly $A \supseteq E_{1/n}$
conversely $\forall x \in A$,
then $f(x) > 0$, so
 $f(x) > 1/n$ for a large n
i.e. $x \in E_{1/n}$ for some n)

$$\begin{aligned}
 \text{Hence } \mu(A) &\leq \sum_{n=1}^{\infty} \mu(E_{1/n}) \\
 &\leq \sum_{n=1}^{\infty} \frac{1}{(1/n)} \int f d\mu \\
 &\leq 0.
 \end{aligned}$$

Hence $\mu(A) = 0$, i.e. $f = 0$ a.e. \square .

Now suppose $f_n \rightarrow f$ a.e.

$$\left(\lim_{n \rightarrow \infty} f_n(x) = f(x) \text{ a.e.} \right),$$

Q: Do we have

$$\int_X f_n d\mu \rightarrow \int_X f d\mu ?$$

Example 1. Let $\mu = d_{(0,1)}$, Leb. measure on $(0,1)$.

Take $\varphi_k = 0$ on $(1/k, 1)$ and $\frac{1}{k}$ on $(0, 1/k)$.

Then $\lim_k \varphi_k = 0$ on $(0, 1)$.

However $\int_{(0,1)} \varphi_k d\mu = 1$

So $\lim_k \int \varphi_k d\mu = 1 \neq \int \lim_k \varphi_k d\mu$

Example 2. Take $f_k = \chi_{[k, k+1]}$

Let $\mu = \mathcal{L}_{[0, +\infty]}$.

Again $f_k \rightarrow 0$ a.e., but

$$\lim_k \int_{(0, \infty)} f_k d\mu = 1 \neq \int \lim_k f_k d\mu.$$

Example 3. Take $g_k = \frac{1}{k} \chi_{[0, k]}$.

$$\mu = \mathcal{L}_{[0, \infty)}.$$

Thm 1.10 (Lebesgue's Monotone Convergence Thm)

Let $f_k, f : X \rightarrow [0, +\infty]$ be measurable.

Assume $f_k(x) \uparrow f(x)$ on $X \setminus N$ with $\mu(N) = 0$

Then

$$\lim_{k \rightarrow \infty} \int_X f_k d\mu = \int_X f d\mu.$$

Pf. Since f_k are monotone increasing,

so are $\int_X f_k d\mu$.

Clearly we have $\int_X f_k d\mu \leq \int_X f d\mu$
(since $f_k \leq f$ a.e.)

Hence $\lim_{k \rightarrow \infty} \int_X f_k d\mu \leq \int_X f d\mu$.

Now we prove the other direction.

Let $0 \leq s \leq f$ be simple. Let $0 < \delta < 1$.

Define: $E_k = \{x \in X \setminus N : f_k(x) \geq \delta \cdot s(x)\}$
 $k=1, 2, \dots$

Since $f_k(x) \uparrow f(x)$ on $X \setminus N$ and $s(x) \leq f(x)$

We have

$$\bigcup_{k=1}^{\infty} E_k = X \setminus N$$

and $E_k \subset E_{k+1}$, $\forall k$.

Now notice that

$$f_k \geq \delta s(x) \chi_{E_k}$$

Taking integration gives

$$\begin{aligned} \int_X f_k d\mu &\geq \delta \int s \chi_{E_k} d\mu \\ &= \delta \int_{E_k} s d\mu \end{aligned}$$

$$= \delta \cdot \sum_{i=1}^N d_i \mu(A_i \cap E_k) \quad \left(s = \sum_{i=1}^N d_i \chi_{A_i} \right)$$

Since $E_k \uparrow X \setminus N$, so $A_i \cap E_k \uparrow A_i \cap (X \setminus N)$
as $k \rightarrow \infty$.

Letting $k \rightarrow \infty$, we see that

$$\begin{aligned} \delta \sum_{i=1}^N d_i \mu(A_i \cap E_k) \\ \rightarrow \delta \sum_{i=1}^N d_i \mu(A_i \cap (X \setminus N)) \\ = \delta \sum_{i=1}^N d_i \mu(A_i) \\ = \delta \cdot \int_X s \, d\mu \end{aligned}$$

Hence

$$\lim_{k \rightarrow \infty} \int_X f_k \, d\mu \geq \delta \int_X s \, d\mu$$

Since δ is arbitrarily taken in $(0, 1)$.

$$\text{Letting } \delta \rightarrow 1, \int_X s \, d\mu \rightarrow \int_X f \, d\mu$$

we have

$$\lim_{k \rightarrow \infty} \int_X f_k \, d\mu \geq \int_X f \, d\mu \quad \square$$

Thm 1.11. (Fatou's Lemma)

Let $f_k: X \rightarrow [0, \infty]$ be measurable, $k \geq 1$.

Then

$$\int_X \liminf_{k \rightarrow \infty} f_k \, d\mu \leq \liminf_{k \rightarrow \infty} \int_X f_k \, d\mu.$$

Pf. Notice that

$$\liminf_{k \rightarrow \infty} f_k(x) = \sup_{k \geq 1} \inf_{j \geq k} f_j(x)$$

Now write $g_k(x) = \inf_{j \geq k} f_j(x)$

$$g(x) = \liminf_{k \rightarrow \infty} f_k(x).$$

Then $g_k \uparrow g$, also g_k are non-negative, measurable.

By Lebesgue's Monotone Convergence Thm

$$\int_X \liminf_{k \rightarrow \infty} f_k(x) \, d\mu = \int_X g \, d\mu$$

$$= \lim_{k \rightarrow \infty} \int_X g_k d\mu$$

$$\leq \lim_{k \rightarrow \infty} \int_X f_k d\mu \quad (\text{since } g_k \leq f_k).$$

Next we prove the linearity of integration.

Prop 1.12: Let $f, g: X \rightarrow [0, +\infty]$ measurable.

Let $\alpha, \beta \geq 0$.

Then
$$\int_X \alpha f + \beta g d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu.$$

pf. First the identity holds if f, g are simple functions.

Next choose $s_k \uparrow f$, $t_k \uparrow g$,
where s_k, t_k are non-negative simple.

Then $\alpha s_k + \beta t_k \uparrow \alpha f + \beta g$

So by the Monotone Convergence Thm

$$\begin{aligned}\int \alpha f + \beta g \, d\mu &= \lim_{k \rightarrow \infty} \int \alpha s_k + \beta t_k \, d\mu \\ &= \lim_{k \rightarrow \infty} \left(\alpha \int s_k \, d\mu + \beta \int t_k \, d\mu \right) \\ &= \alpha \int f \, d\mu + \beta \int g \, d\mu.\end{aligned}$$

□

Now we are ready to define the integration of general measurable functions.

Def. Let $f: X \rightarrow [-\infty, \infty]$ be measurable.

Then we define

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu$$

if one of $\int_X f^+ \, d\mu$, $\int_X f^-$ is finite.

where $f^+ = \max\{f, 0\}$, $f^- = \max\{0, -f\}$

Def. We say a measurable function f is integrable if $\int_X f^+ d\mu < \infty$ and $\int_X f^- d\mu < \infty$.

(Notice $|f| = f^+ + f^-$. Hence by Prop 1.12,

$$\int |f| d\mu = \int f^+ d\mu + \int f^- d\mu,$$

so f is integrable $\Leftrightarrow \int_X |f| d\mu < \infty$.

Prop 1.13. Let f, g be integrable and $\alpha, \beta \in \mathbb{R}$

Then $\alpha f + \beta g$ is integrable and

$$\int_X \alpha f + \beta g d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu.$$

Pf. We first prove $f+g$ is integrable and

$$\int f+g d\mu = \int f d\mu + \int g d\mu$$

Since $|f+g| \leq |f| + |g|$, so

$$\int |f+g| d\mu \leq \int |f| d\mu + \int |g| d\mu < \infty$$

Hence $f+g$ is integrable.

Now we prove $\int f+g d\mu = \int f d\mu + \int g d\mu$.

Notice that

$$\begin{aligned} f+g &= (f+g)^+ - (f+g)^- \\ &= (f^+ - f^-) + (g^+ - g^-) \end{aligned}$$

Hence

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+.$$

Taking integration on both sides, we obtain

$$\begin{aligned} \int (f+g)^+ d\mu + \int f^- d\mu + \int g^- d\mu &= \int (f+g)^- d\mu \\ &\quad + \int f^+ d\mu + \int g^+ d\mu \end{aligned}$$

from which we obtain

$$\begin{aligned} \int (f+g)^+ d\mu - \int (f+g)^- d\mu &= \int f^+ d\mu - \int f^- d\mu \\ &\quad + \int g^+ d\mu - \int g^- d\mu \end{aligned}$$

$$\text{i.e. } \int f+g d\mu = \int f d\mu + \int g d\mu.$$

Next we show $c \int f d\mu = \int cf d\mu$, $\forall c \in \mathbb{R}$.

If $c > 0$, then it follows from the def of integration of meas. function since $(cf)^+ = cf^+$
 $(cf)^- = cf^-$.

If $c < 0$, it suffices to show

$$-\int f d\mu = \int -f d\mu.$$

Again it follows from the def. \square

Thm 1.14 (Lebesgue's dominated convergence Thm).

Let $f, f_k : X \rightarrow [-\infty, \infty]$ be measurable such that

$$f_k(x) \rightarrow f(x) \text{ a.e. as } k \rightarrow \infty.$$

Moreover, suppose \exists an integrable g such that

$$|f_k(x)| \leq g(x) \text{ a.e. for all } k \in \mathbb{N}.$$

Then

$$\lim_{k \rightarrow \infty} \int_X f_k d\mu = \int_X f d\mu.$$

Pf. First $|f(x)| = \lim_{k \rightarrow \infty} |f_k(x)| \leq g(x)$ a.e

So f is integrable.

Now let us apply Fatou's lemma to the sequence $2g - |f_k - f|$, $k=1, 2, \dots$

$$(|f_k - f| \leq |f_k| + |f| \leq 2g \text{ a.e.})$$

We have

$$\begin{aligned} \lim_{k \rightarrow \infty} \int 2g - |f_k - f| \, d\mu &\geq \int \lim_{k \rightarrow \infty} (2g - |f_k - f|) \, d\mu \\ &\geq \int 2g \, d\mu. \quad (***) \end{aligned}$$

$$\begin{aligned} \text{However, } \lim_{k \rightarrow \infty} \int 2g - |f_k - f| \, d\mu &= \int 2g \, d\mu + \lim_{k \rightarrow \infty} (-1) \int |f_k - f| \, d\mu \\ &= \int 2g \, d\mu - \overline{\lim}_{k \rightarrow \infty} \int |f_k - f| \, d\mu \\ &\geq \int 2g \, d\mu \quad (\text{by } (***)), \end{aligned}$$

from which we have $\overline{\lim}_{k \rightarrow \infty} \int |f_k - f| \, d\mu \leq 0$.

Hence $\lim_{k \rightarrow \infty} \int |f_k - f| d\mu = 0.$

So $\lim_{k \rightarrow \infty} \left| \int f_k d\mu - \int f d\mu \right|$

$$\leq \lim_{k \rightarrow \infty} \int |f_k - f| d\mu = 0.$$

Therefore

$$\lim_{k \rightarrow \infty} \int f_k d\mu = \int f d\mu. \quad \square$$