Definition (Informal)

A function is said to be continuous at "$x=c$" if $\lim_{x \to c} f(x) = f(c)$.

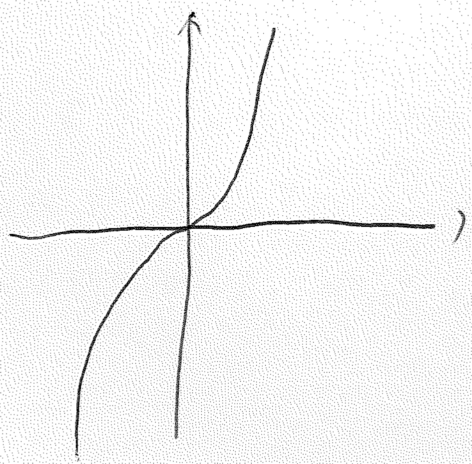
We say $f(x)$ is continuous at "$x=c$", and we say $f(x)$ is continuous at $\underline{\underline{(a,b)}}$
$a<x<b$

if $f(x)$ is continuous at every point $x \in (a,b)$.

(1) continuous at one point → local

(2) continuous on a interval → global

example 2.



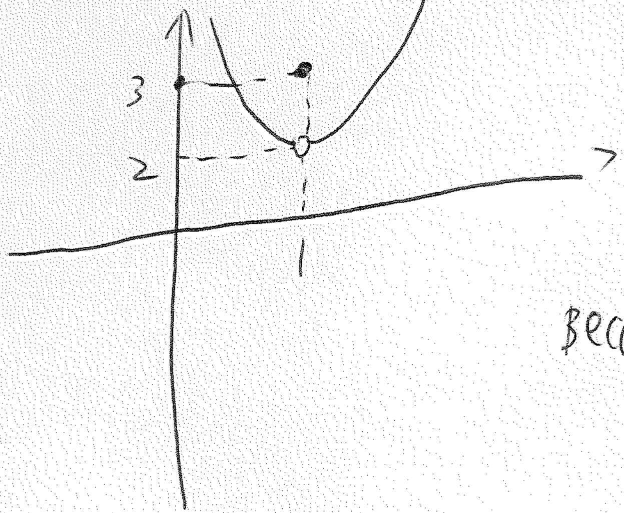
$f(x)=x^3$ Is this function continuous at $x=0$?

$$\lim_{x \to 0} x^3 = 0 = f(0) = 0$$

Continuous

A New Concept: Continuity

Do you remember this example:



→ Is this function

$$f(x)=\begin{cases}(x-1)^2+2, & x\neq 1\\ 3, & x=1\end{cases}$$

continuous on $[0,2]$?

Because: When $x=1$, $\lim_{x\to 1} f(x) \neq f(1)$.

Not continuous

In this example, we have talked about the limit has nothing to do with $f(c)$.

$\lim_{x\to c} f(x)$

It only concerns points around the point "c". So in which case

$f(c)$ and $\lim_{x\to c} f(x)$ have a relationship

Answer: when $f(x)$ is continuous at "c"?

Example 3.

$$f(x)=\begin{cases} \frac{\sin x}{x}, & x\neq 0 \\ a, & x=0. \end{cases}$$

For which value of a, we can know f(x) is continuous?

"a" = 1. $\lim_{x\to 0}\frac{\sin x}{x}=1$ (Remember this limit)

We have known the concept "continuity" is strongly to

"$\lim_{x\to c} f(x)=f(c)$". How to give the formal definition?

(ε-δ language)

For any $\varepsilon>0$, $\exists\,\delta>0$ such that, if

$|x-a|<\delta \Rightarrow$ we have $|f(x)-f(a)|<\varepsilon$ for any $x\in\{x \mid |x-a|<\delta\}$

What is the difference between "continuous" and "limit"?

$\lim_{x\to a} f(x)=L$: For any $\varepsilon>0$, $\exists\,\delta>0$, such that if

$0<|x-a|<\delta \Rightarrow$ we have $|f(x)-L|<\varepsilon$ for any $x\in\{x \mid 0<|x-a|<\delta\}$

when we consider "f(x) is continuous at "x=c"", we need to concern "$f(x)|_{x=c}$". the value of f(c) is important for the continuity!

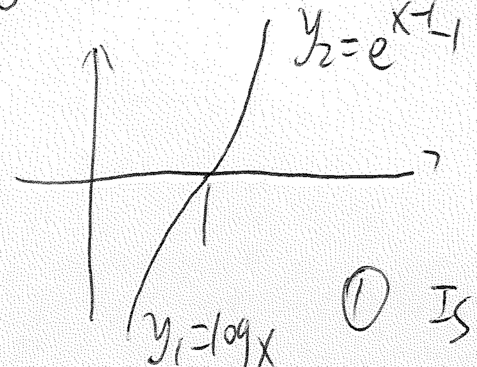
when we consider $\lim_{x\to c} f(x)$, f(c) is not important at all.

We know f(x) is continuous at "x=c" if $\lim_{x\to c} f(x) = f(c)$

Do you remember, $\lim_{x\to c} f(x) = \underbrace{\lim_{x\to c^+} f(x)}_{\text{right limit}} = \underbrace{\lim_{x\to c^-} f(x)}_{\text{left limit}}$, so we can prove the

continuity of f(x) at one point, by the left limit and right limit.

Example



$$f(x)=\begin{cases}\log x, & 0<x<1,\\ e^{x-1}-1, & 1\le x.\end{cases}$$

① Is this function continuous at "x=1"?

② Try using ① $\lim_{x\to 1} f(x)=f(1)$ ② $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^-} f(x)=f(1)$

$\underline{\lim_{x \to 1} f(x) = 0}$. $x_n = \begin{cases} 1+\frac{1}{n}, & n \text{ is odd} \\ 1-\frac{1}{n}, & n \text{ is even} \end{cases}$, jump!

If we use left limit \ right limit, very clear.

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} e^{x-1} - 1 = 0$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \log x = 0$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$$

Why "$\lim_{x \to 1^+} \underline{\underline{e^{x-1} - 1}} = 0$". We think "$\underline{e^{1-1} - 1 = 0}$"

continuous

$\lim_{x \to c} f(x) = f(c)$ if $f(x)$ is continuous

Basic rules (

① If $f(x)$ and $g(x)$ are continuous at $x=c$, then $f(x) \pm g(x)$, $\frac{f(x)}{g(x)}$ $(g(x) \neq 0)$ $f(x)g(x)$ are continuous at $x=c$.

② Polynomials and exponential functions are continuous everywhere

③ Trigonometric and logarithmic functions are continuous everywhere

④ Let $g(u)$ be a function of u and $u=f(x)$. If $g(u)$ is continuous with respect to u and the limit $\lim_{x \to a} f(x)$ exists.

Then $\lim_{x \to a} (g \circ f)(x) = \lim_{x \to a} g(f(x)) = g(\lim_{x \to a} f(x))$

We interchange limit and function.

$$\boxed{\lim_{x \to a} \sum_{n=0}^{\infty} f(n,x) = \sum_{n=0}^{\infty} \lim_{x \to a} f(n,x)}$$

$$\sum_{n=1}^{\infty} \frac{x}{n} = \infty \quad \text{if } x \neq 0$$

$$\lim_{x \to 0} \sum_{n=1}^{\infty} \frac{x}{n} = \infty \qquad \sum_{n=1}^{\infty} \lim_{x \to 0} \frac{x}{n} = \sum_{n=1}^{\infty} 0 = 0$$

$\downarrow$

$x \neq 0$