1.4 mathematical induction

If we want $P(n)$ is right for any $n \geq 1$, $n \in \mathbb{Z}^+$.

(1) We prove $P(1)$ is right

(2) We prove $P(n+1)$ is right, if $P(n)$ is right, $n \geq 1$

(3) We complete the proof

Example 1.4.3: Let $P(n)$ be the statement that $\sum_{r=1}^{n} \cos r\theta = \dfrac{\cos(\frac{(n+1)\theta}{2})\sin(\frac{n\theta}{2})}{\sin\frac{\theta}{2}}$, where $n \in \mathbb{Z}^+$.

Use mathematical induction to prove this identity.

(1) $P(1) = \cos\theta = \dfrac{\cos\theta \sin\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cos\theta$

(2) $P(n) \Rightarrow P(n+1)$, $n \geq 1$ $\left(P(n) = \sum_{r=1}^{n} \cos r\theta = \dfrac{\cos(\frac{(n+1)\theta}{2})\sin(\frac{n\theta}{2})}{\sin\frac{\theta}{2}} \right)$.

For $P(n+1) = \sum_{r=1}^{n+1} \cos r\theta = \sum_{r=1}^{n} \cos r\theta + \cos(n+1)\theta$

$$= \frac{\cos(\frac{n+1}{2}\theta)\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}} + \cos(n+1)\theta = \frac{\cos(\frac{n+1}{2}\theta)\sin\frac{n}{2}\theta + \cos(n+1)\theta\sin\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$2\cos\alpha\cos\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

$\cancel{\cos(\frac{n+1}{2}\theta)\sin\frac{n\theta}{2} + \cos[(n+1)\theta]}$

$$= \frac{\frac{1}{2}\left[\sin\frac{2n+1}{2}\theta - \sin\frac{\theta}{2}\right] + \frac{1}{2}\left[\sin\frac{2n+3}{2}\theta - \sin\frac{2n+1}{2}\theta\right]}{\sin\frac{\theta}{2}}$$

$$= \frac{\frac{1}{2}\left[\sin\frac{2n+3}{2}\theta - \sin\frac{\theta}{2}\right]}{\sin\frac{\theta}{2}}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$= \frac{\cos\left(\frac{n+2}{2}\theta\right)\sin\left(\frac{n+1}{2}\theta\right)}{\sin\frac{\theta}{2}}$$

$\therefore$ $p(k+1)$ is true.

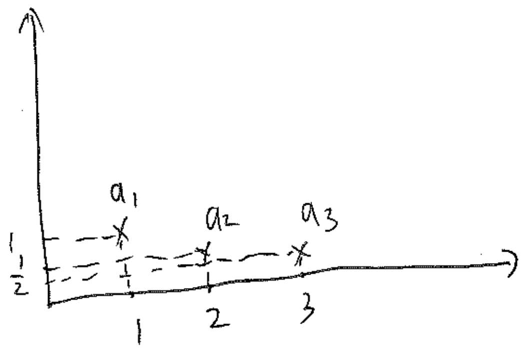
1.5 sequences of Real numbers

① $a_1=1,\ a_2=3,\ a_3=5,\ a_4=7, \cdots\cdots, a_n=2n-1$ (Arithmetic sequence)

② $a_1=1,\ a_{n+1}=a_n^2+2$ for $n \geq 1$, (Recursive sequence)

③ $a_1=3,\ a_2=3.1,\ a_3=3.14,\ a_4=3.145,\ a_5=3.1456,\ a_n=\underbrace{3.\cdots\cdots}_{n \text{ random number}}$

④ $a_1=1,\ a_2=\frac{1}{2},\ a_3=\frac{1}{3}, \cdots\cdots, a_n=\frac{1}{n}$



$\lim\limits_{n\to\infty} a_n=0$

Limits of sequences

2.1 Definition (informal)

Let $\{a_n\}$ be a sequence of real numbers.

If n is getting larger and larger, a_n is getting closer and closer to L.

We say "L" is the limit of the sequence $\{a_n\}$ and we denote it by $\lim_{n\to\infty} a_n = L$.

"Does the limit always exist"

A: Of course not. To determine whether the limit exists or not is an important question.

Example: $a_n = 2^{n-1}$, $\lim_{n\to\infty} a_n = \infty$ (we say a_n diverges)

Two concepts: (informal)

① Converge: $\lim_{n\to\infty} a_n = L$ (a_n converges to L)

② Diverge: $\lim_{n\to\infty} a_n = \begin{cases} +\infty \\ -\infty \end{cases}$ (a_n diverges)

Neither "converge" nor "diverge": $\begin{cases} a_n = 1, & n \text{ is even} \\ a_n = -1, & n \text{ is odd} \end{cases}$

Impossible to have a limit

$\varepsilon-\delta$ language, for sequence (ε-N definition)

Let $\{a_n\}$ be a sequence of real number and "$L \in \mathbb{R}$".

"L" is said to be the limit of the sequence $\{a_n\}$

"if"

$\forall \varepsilon > 0, \ \exists N \in \mathbb{Z}^+, \ s.t. \ |a_n - L| < \varepsilon, \ \forall n \geq N$

"$a_n = \frac{1}{n}$", $n \geq 1$, we know $\lim_{n \to \infty} a_n = 0$

① $|a_n - 0| < \frac{1}{100}$, what is the range of n? $n > 100$

② $|a_n - 0| < \frac{1}{1000}$, $n > 1000$

"$|a_n - 0| = 0 \Rightarrow n = +\infty$" (Informal)

Example $P(n) = 0.\underbrace{9999\cdots}_{\text{The number of "9" is } n}$

$P(1) = 0.9$

$P(3) = 0.999$

$\lim_{n \to \infty} P(n) = 1$

① $|p_n - 1| \leq 0.01$, $n \geq 2$, $\varepsilon = 0.01$

② $|p_n - 1| \leq 0.001$, $n \geq 3$, $\varepsilon = 0.001$

③ $|p_n - 1| \leq 10^{-k}$, $n \geq k$

$\lim_{n\to\infty} p_n = 1 \Rightarrow 1 = 0.\dot{9}\cdots\cdots$

Limitations of some sequences

(1) $a_n = k$, $\lim_{n\to\infty} a_n = k$ (The limit of a constant sequence is a constant)

(2) $-1 < a < 1$, $\lim_{n\to\infty} a^n = 0$

(3) Suppose that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M \Rightarrow \begin{cases} \text{if } a_n \leq b_n, \text{ then } L \leq M \\ \text{if } a_n \geq 0, \text{ then } \lim_{n\to\infty} \sqrt{a_n} = \sqrt{L}. \end{cases}$

(4) if $a_n < b_n$, then $L < M$. "✓"

$\frac{1}{n+1}$ $\frac{1}{n}$ 0 0 "X" ⟶ This!

Algebraic Properties of Limits

Theorem 2.2.1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers

if $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$, then

1) $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$

Consider

$$\begin{cases} 0 = \lim_{n\to\infty} a_n + b_n \\ a_n = -b_n, \quad |a_n| = |b_n| = 1 \end{cases}$$

$$\Rightarrow \lim_{n\to\infty} a_n = 1 \text{ or } -1$$

$$a_n = \begin{cases} 1, & n \text{ is odd} \\ -1, & n \text{ is even} \end{cases}$$

$$b_n = \begin{cases} 1, & n \text{ is even} \\ -1, & n \text{ is odd} \end{cases}$$

② $\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L - m$

③ $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = Lm$

$\begin{cases} a_n = n \\ b_n = \frac{1}{n} \end{cases} \Rightarrow |L| \& |m| < \infty$

④ $m \neq 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{m}$.

Example 2.2.1

$\lim_{n\to\infty} \frac{2}{n} + 3 = 3$

$\parallel$

$\lim_{n\to\infty} \frac{2}{n} + 3$

$\parallel$

3

Example 2.2.2

$$\lim_{n\to\infty} \frac{n^2+3}{2n^2-4n} = \frac{1}{2}$$

$$\lim_{n\to\infty} \frac{n^3+2n}{2n^2+1} = \infty$$

$$\lim_{n\to\infty} \frac{n+1}{n^2} = 0$$

If $p(x)$ and $q(x)$ are polynomials.

$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ with $a_m \neq 0$

$q(x) = b_k x^k + b_{k-1} x^{k-1} + \cdots + b_0$

then
$$\lim_{n\to\infty} \frac{p(n)}{q(n)} = \begin{cases} \pm\infty, & \text{if } m > k \\ \dfrac{a_m}{b_k}, & \text{if } m = k \\ 0, & \text{if } m < k \end{cases}$$

Example 2.2.3

$$\lim_{n\to\infty} \frac{3n-1}{\sqrt{4n^2+2n}} = \frac{3}{2}$$

$$\lim_{n\to\infty} \sqrt{n+1}-\sqrt{n} = 0$$

$$\lim_{n\to\infty} \frac{2^n}{n} = \infty$$

2.3 Monotonic sequence Theorem

Let $\{a_n\}$ be a sequence of real numbers

(1) $\{a_n\}$ is said to be bounded above if $\exists M \in \mathbb{R}$ s.t. $a_n \leq M$ (called <u>an upper bound</u>)

(2) $\{b_n\}$ is said to be bounded below if $\exists M \in \mathbb{R}$, s.t. $b_n \geq M$ (called <u>a lower bound</u>)

(3) $\{a_n\}$ is said to be bounded $\Leftrightarrow$ bounded above and below

(4) a_n is said to be monotonic increasing if $a_{n+1} \geq a_n$, $\forall n \in \mathbb{Z}^+$

(5) a_n is said to be monotonic decreasing if $a_{n+1} \leq a_n$, $\forall n \in \mathbb{Z}^+$

(6) a_n is said to be monotonic if either increasing or decreasing

Relationship between monotonic & ~~convergent~~ bounded

① Monotonic does not imply convergent $a_n = n+1$

② bounded does not imply convergent $a_n = \begin{cases} 1, & n \text{ is odd} \\ -1, & n \text{ is even} \end{cases}$

③ monotonic & bounded imply convergent

Theorem 2.3.1 (Monotone Convergence Theorem)

If $\{a_n\}$ is bounded above (below) and monotonic increasing (decreasing),

then $\lim_{n\to\infty} a_n$ exists:

Example 2.3.2

Let $\{a_n\}$ be a sequence of positive real numbers defined by

$a_1 = 1$ and $a_{n+1} = 1 + \frac{a_n}{1+a_n}$ $(n \geq 1)$

Does $\lim_{n\to\infty} a_n$ exist?

① $\{a_n\}$ is bounded

② $\{a_n\}$ is monotonic

(i) : ① $a_2 - a_1 = \frac{1}{2}$

② $a_{k+2} - a_{k+1} = \frac{a_{k+1} - a_k}{(1+a_{k+1})(1+a_k)} \geq 0$

(ii) $\{a_n\}$ is bounded : $|a_{n+1}| \leq 1 + \left|\frac{a_n}{1+a_n}\right| \leq 2$

What is the limit $\lim_{n\to\infty} a_n$: $a_{n+1} = 1 + \frac{a_n}{1+a_n}$, we assume $\lim_{n\to\infty} a_n = A$

So we have $A = 1 + \frac{A}{1+A} \Rightarrow A + A^2 = (1+A) + A \Rightarrow A^2 - A - 1 = 0.$

$A = \frac{1+\sqrt{5}}{2}$ or $\frac{1-\sqrt{5}}{2}$ (reject)

$\lim_{n\to\infty} (1+\frac{1}{n})^n = e$: " $\lim_{x\to n} f(x) = f(\lim_{x\to n} x)$ if f is continuous "

$\log e = \lim_{n\to\infty} \log(1+\frac{1}{n})^n = \lim_{n\to\infty} n \log(1+\frac{1}{n}) = 1$

Example 2.2.3

Let $\{a_n\}$ be a sequence of real numbers defined by "$a_n = (1+\frac{1}{n})^n$" Prove that

(1) $\{a_n\}$ is monotonic increasing

(2) $\{a_n\}$ is bounded above

$$a_n = (1+\frac{1}{n}) = \sum_{r=0}^{n} C_r^n \frac{1}{n^r} = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}\frac{1}{n^r}$$

$$- \quad + \cdots + \frac{n(n-1)\cdots(n-n+1)}{n!}\frac{1}{n^n}.$$

$$= 1+1+\frac{1}{2!}(1-\frac{1}{n}) + \frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n}) + \cdots + \frac{1}{r!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{r-1}{n})$$

$$+ \cdots + \frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})$$

Similarly:

$$a_{n+1} = 1+1+\frac{1}{2!}(1-\frac{1}{n+1}) + \frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1}) + \cdots + \underbrace{\frac{1}{r!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{r-1}{n+1})}$$

$$+ \cdots$$

Make an explanation: $\frac{(n+1)(n+1-1)\cdots(n+1-r+1)}{r!}\frac{1}{(n+1)^r} = \frac{1}{r!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{r-1}{n+1})$

$$> \frac{1}{r!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{r-1}{n})$$

So $a_{n+1} \geq a_n$

$$a_n = 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\cdots+\frac{1}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left(1-\frac{r-1}{n}\right)$$

$$+\cdots\cdots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\cdots\left(1-\frac{n-1}{n}\right)$$

$\leq 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots\cdots+\frac{1}{r!}+\cdots\cdots+\frac{1}{n!}$ [we use $1-\frac{1}{n}<1$, $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)<1$, $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)<1$ here]

$\leq 1+1+\frac{1}{2}+\frac{1}{2^2}+\cdots\cdots+\frac{1}{2^{n-1}}$ (we use $\frac{1}{2^{n-1}}\leq\frac{1}{n!} \Rightarrow n!\leq 2^{n-1}$ (here))

$$\leq 1+\frac{1}{1-\frac{1}{2}}$$

$$=3$$

$\{a_n\}$ is monotonic and bounded

$$a_n=\sum_{r=0}^{n} a_{n,r}, \quad \lim_{n\to\infty} a_{n,r}=\frac{1}{r!}$$

$$\lim_{n\to\infty} a_n=\sum_{r=0}^{\infty}\lim_{n\to\infty} a_{n,r}=\sum_{r=0}^{\infty}\frac{1}{r!}=e$$