Three properties of a set

① Well-definedness:

→ {Everyone is tall in this class} → not a set, what is tall

→ {Everyone $\geq$ 150 cm in this class}

② Unorderness

$\{a, b, c\}$ and $\{a, c, b\}$

③ Uniqueness

$\{a, a, b, c\}$ and $\{a, b, c\}$

Example 1.1.1.

$S = \{1, 2, 3\}$

Elements of S: 1, 2, 3 : $1, 2, 3 \in S$

$\in$ belongs to

$T = \{1, 2, 3, 4\}$, $S \subseteq T$

$\subseteq$ is a subset : Every element in S is an element in T

$\mathbb{Z}^+$: Set of all Positive integers

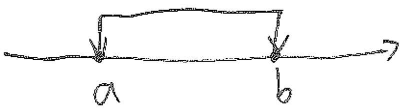
$\mathbb{Z}$: set of all integers

$\mathbb{R}$: Set of all real numbers

ϕ : empty set, i.e. $\phi = \{ \}$

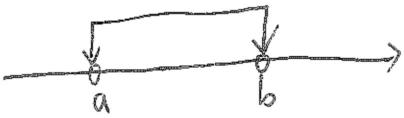
$\{\phi\} \rightarrow$ not empty set

$[a,b]$: set of all real numbers "x" such that $a \leq x \leq b$

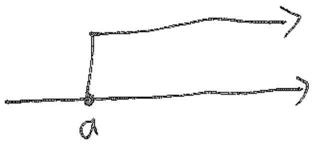


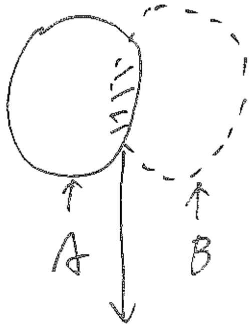
"included" "not included"

(a,b) : set of all real numbers "x" such that $a < x < b$

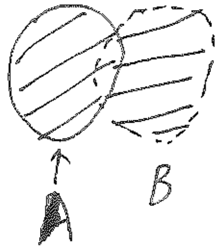


$[a,\infty)$: Set of all real numbers "x" such that $a \leq x$

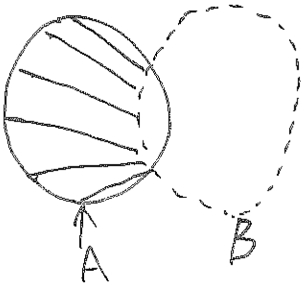




$A \cap B$: Intersection of A and B

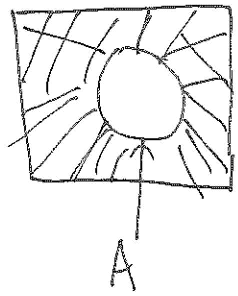


$A \cup B$: Union of A and



Relative complement of B in A : $A \setminus B$

Just think: $A - B$



Complement of A : A

Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{4\}$

$A \cap B = \{2\}$ $\quad A \cap C = \phi$ $\quad A \cup B = \{1,2,3\}$

$A \setminus B = \{1\}$ $\quad B \setminus A = \{3\}$

$A \cup B \cup C \longrightarrow$ biggest set we consider

$C^c = \{1,2,3\}$

Example 1.1.5

$\{x \mid x^2 > 1\} = \{x \mid \text{"}x>1\text{" or "}x<-1\text{"}\} = (-\infty,-1) \cup (1,\infty) = \underbrace{R \setminus [-1,1]}_{\text{set of all real numb... real numbers } x \text{ ...}}$

$x>1 \uparrow (1,+\infty) \Updownarrow (1,\infty)$

$x<-1 \uparrow (-\infty,-1) \Downarrow ? \; \underline{(\infty,-1)} \times$ Can not ingore "$-$"

All real numbers : $\underset{\substack{\uparrow \\ \text{for all}}}{\forall}\, x \underset{\substack{\uparrow \\ \text{belong to}}}{\in} R \leftarrow$ the set of real numbers

$\forall$: ~~for~~ all

$\exists$: there exists (at least one)

$\exists!$: there exists unique

$\Rightarrow$: implies

$\Leftrightarrow$: if and only if

s.t. : such that

Example 1.1.6

$\forall\, y \in (0,\infty),\ \exists\, X \in \mathbb{R}$ s.t. $X^2 = y$.

Answer: For all real number y belongs to 0 to $+\infty$, there exists (at leas

real number X such that $X^2 = y$.

$\forall\, y \in (0,\infty),\ \exists!\, X \in (0,\infty)$, s.t. $X^2 = y$.

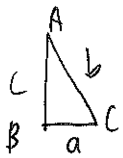
Answer: For all real number y belongs to 0 to $+\infty$, there exists unique X

"0" to "$+\infty$" such that $X^2 = y$.

Example 1.1.8 :

$\Leftrightarrow$ if and only if

Two lines have no common points $\Leftrightarrow$ They are parellel to each other in $\mathbb{R}^2$

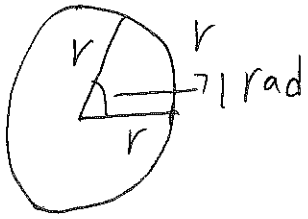


$$b^2 = a^2 + c^2 \Longleftrightarrow \angle ABC = 90^\circ = \frac{\pi}{2}$$

$\uparrow$ Radian.

Radian

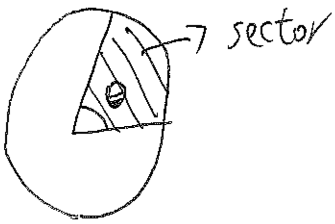
Arc length = radius



$$2\pi \text{ rad} = 360^\circ$$

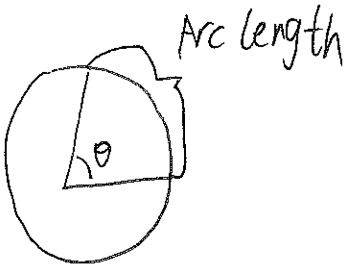
$$\pi \text{ rad} = 180^\circ$$

$$90^\circ = \frac{\pi}{2} \text{ rad}$$



$$\frac{\text{Area of the sector}}{\text{Area of the ball}} \propto \theta / 2\pi$$

$$A = \frac{1}{2} r^2 \theta$$



$$\frac{\text{Arc length}}{\text{Ball}} \propto \frac{\theta}{2\pi}$$

$$\Rightarrow \frac{l}{2\pi r} = \frac{\theta}{2\pi}$$

$$\Rightarrow l = r\theta$$

Trigonometric identities

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Double angle formula

$\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$$\begin{cases} \tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\ \tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} \end{cases}$$

Product to sum formula

$$\begin{aligned} 2\cos\alpha\cos\beta &= \cos(\alpha+\beta) + \cos(\alpha-\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= 2\cos\alpha\cos\beta \end{aligned}$$

$-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$

$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$

$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$

Sum to product formula:

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\parallel$$

$$\sin\left(\frac{A+B}{2}+\frac{A-B}{2}\right)+\sin\left(\frac{A+B}{2}-\frac{A-B}{2}\right)$$

$$= \sin\frac{A+B}{2}\cos\frac{A-B}{2}+\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$+\sin\frac{A+B}{2}\cos\frac{A-B}{2}-\cos\left(\frac{A+B}{2}\right)\sin\frac{A-B}{2}$$

$$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\begin{cases} A = \dfrac{A+B}{2}+\dfrac{A-B}{2} \\ B = \dfrac{A+B}{2}-\dfrac{A-B}{2} \end{cases}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

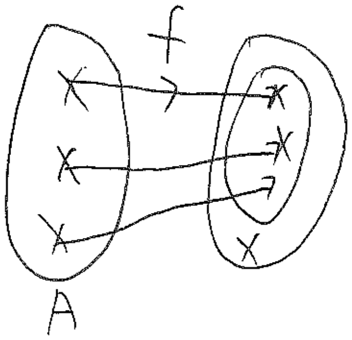
$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

1.3 Functions

Function: A function a rule that assigns to each elemet in a se
exactly one elemet in a set B

Set A: domain

Set B: codomain



range(f) $\subseteq B$

$\underline{u: R \to R}$

$\underline{\text{range}(u)}$

Example 1.3.1

if 1) $f: \mathbb{R} \to \mathbb{R}$ defined by $\begin{cases} f(x) = x^2 \\ f(x) = x^2 + 1 \end{cases}$

$\downarrow$

$\mathbb{R}^+ \cup \{0\}$

$[1, +\infty)$

Example 1.3.2

If $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 4$

$[4, +\infty)$

Example 1.3.3

If $f(x) = \frac{2x}{x^2 - 7x}$, what is the (maximum) domain of f?

$x^2 - 7x \neq 0 \Rightarrow x \neq 0$ and $x \neq 7$

Domain of f: $\{x \in \mathbb{R} : x \neq 0, x \neq 7\}$

$= (-\infty, 0) \cup (0, 7) \cup (7, \infty)$

$\mathbb{R} \setminus \{0, 7\}$

Example: 1.3.4

If $f(x)=\sqrt{x^2-4x+3}$, find the (maximum) domain of f

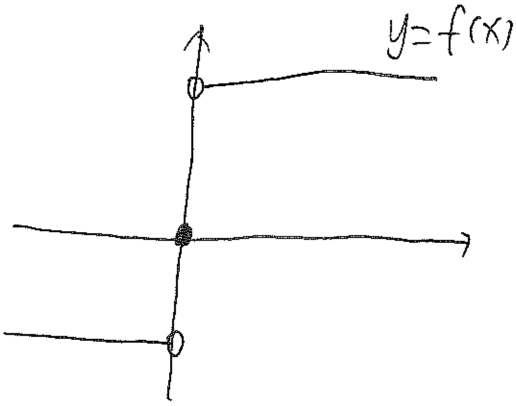
Domain of $f=\{x\in\mathbb{R} : x\leq 1 \text{ or } x\geq 3\}$

Exercise 1.3.1

If $f(x)=\dfrac{1}{\sqrt{x^2-4x+3}}$, find the (maximum) domain of f.

$R\setminus[1,3]$

Introduce some basic functions

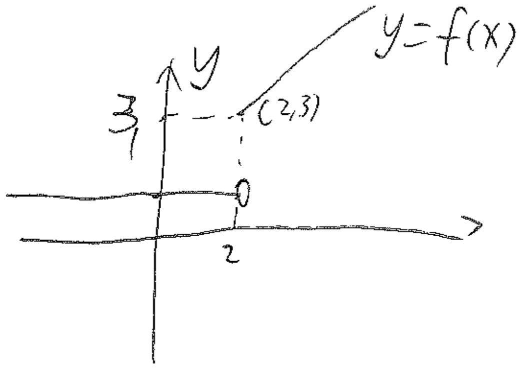


$$f(x)=\begin{cases} 1 & \text{if } x>0, \\ 0 & \text{if } x=0, \\ -1 & \text{if } x<0. \end{cases}$$

piecewise defined function

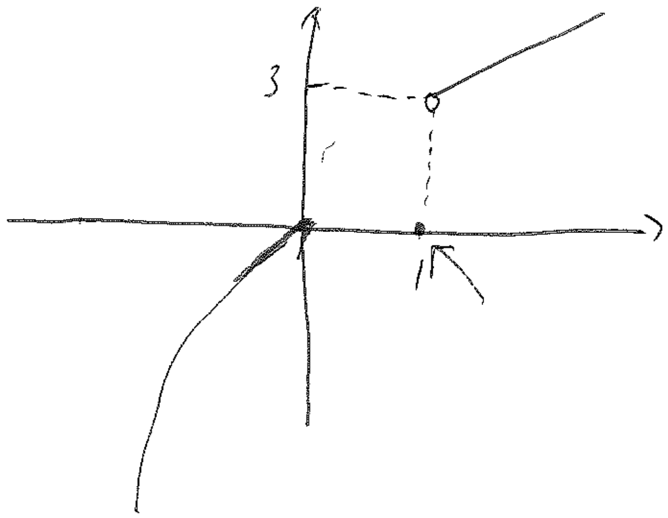
Example 1.3.6

$$f(x) = \begin{cases} x+1, & \text{if } x \geq 2, \\ 1, & \text{if } x < 2. \end{cases}$$



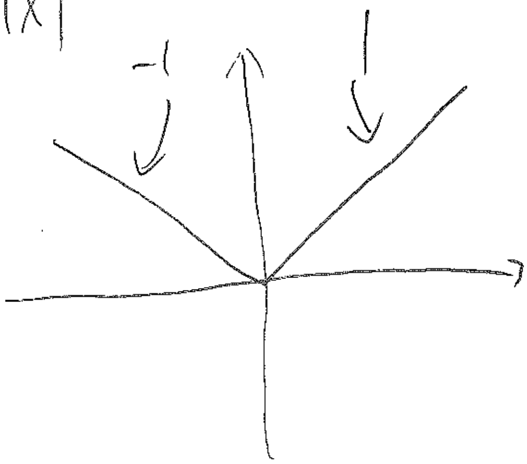
Exercise 1.3.2

Sketch the graph of $f(x) = \begin{cases} 2x+1, & \text{if } x > 1 \\ 0, & \text{if } 0 \leq x \leq 1 \\ -x^2, & \text{if } x < 0 \end{cases}$

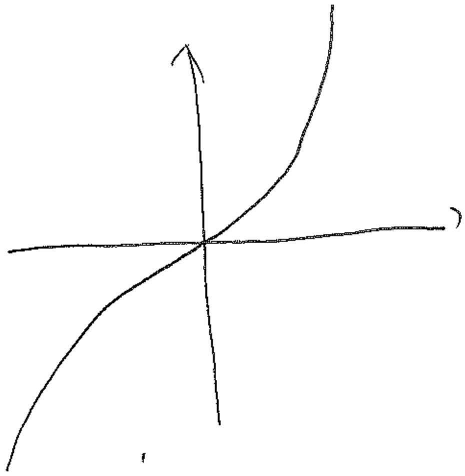


0

$y=|x|$



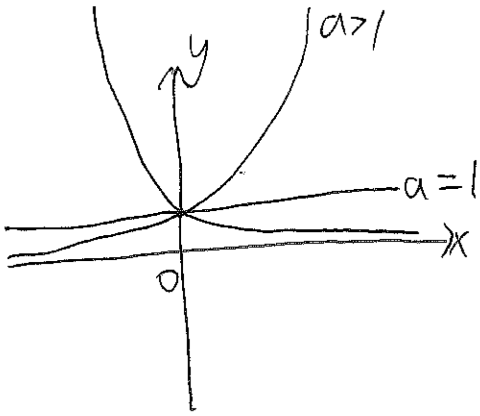
$y=x^3$



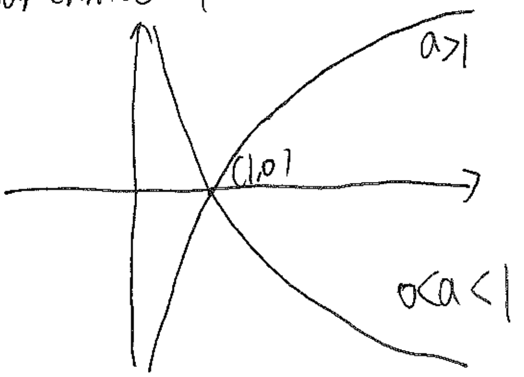
Exponential and Logarthmic Functions:

① $y = a^x$

$a > 1$



② Logarthmic Functions $y = \log_a x$ with "$a > 1$" or "$0 < a < 1$"



1) $\log_a M + \log_a N = \log_a MN$

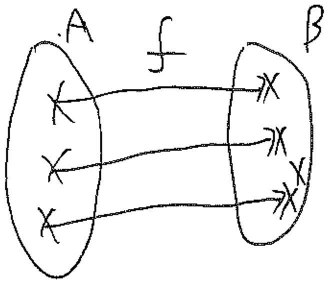
2) $\log_a M^n = n \log_a M$

3) $\log_a x = \frac{\log_b x}{\log_b a}$ (Change of base)

4) a^x and $\log_a x$ are inverse to each other

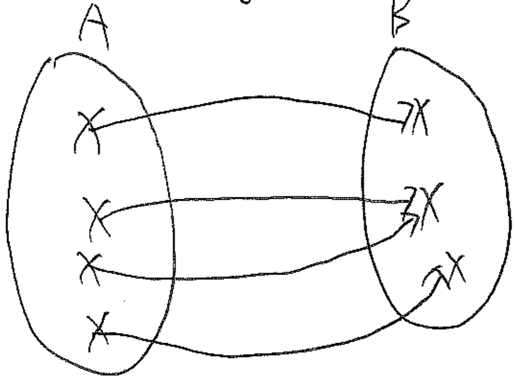
injective and Surjective Functions

Injective: Every $y \in range(f)$ comes from exactly one $x \in A$

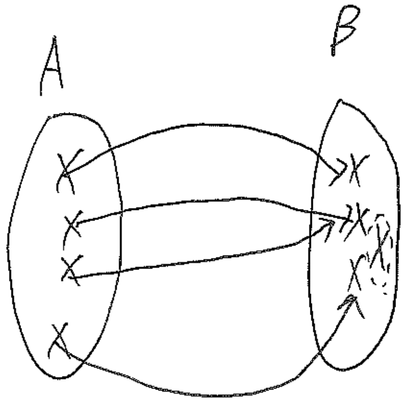


Surjective: Every $y \in B$ comes from at least one $x \in A$

Bijective: Injective and surjective



surjective but not injective



Neither injective nor surjective

Definition 1.3.1