Integration of Trigonometric function

① $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{1}{\cos x} d\cos x = -\ln|\cos x| + c$

② $\int \frac{1}{\tan x} dx = \int \frac{\cos x}{\sin x} dx = \int \frac{1}{\sin x} d\sin x = \ln|\sin x| + c$

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③ $\int \frac{1}{\sin x} dx$ ~~(Double angle identities)~~ (Weierstrass substitution)

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2\tan(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}$$

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{1} = \frac{\cos^2(\frac{x}{2})-\sin^2(\frac{x}{2})}{\cos^2(\frac{x}{2})+\sin^2(\frac{x}{2})} = \frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}$$

So we have

$$\int \frac{1}{\sin x} dx = \int \frac{1+\tan^2(\frac{x}{2})}{2\tan(\frac{x}{2})} dx$$

Here $\frac{d\tan(\frac{x}{2})}{dx} = \frac{1}{2}\frac{d\tan(\frac{x}{2})}{d\frac{x}{2}} = \frac{1}{2}\frac{1}{\cos^2(\frac{x}{2})}$

$$\Rightarrow dx = 2\cos^2(\tfrac{x}{2})\, d\tan(\tfrac{x}{2}) = 2\frac{\cos^2(\frac{x}{2})}{\sin^2(\frac{x}{2})+\cos^2(\frac{x}{2})} d\tan(\tfrac{x}{2})$$

$$= 2\frac{1}{\tan^2(\frac{x}{2})+1} d\tan(\tfrac{x}{2}).$$

So we have $\int \frac{1}{\sin x} dx = \int \frac{1+\tan^2(\frac{x}{2})}{2\tan(\frac{x}{2})} dx = \int \frac{1}{\tan(\frac{x}{2})} d\tan\frac{x}{2} = \ln|\tan(\frac{x}{2})| + C.$

④ $\int \frac{1}{\cos x} dx = \int \frac{1+\tan^2(\frac{x}{2})}{1-\tan^2(\frac{x}{2})} dx = \int \frac{1+\tan^2(\frac{x}{2})}{1-\tan^2(\frac{x}{2})} \frac{2}{\tan^2(\frac{x}{2})+1} d\tan(\frac{x}{2})$

$$= \int \frac{2}{1-\tan^2(\frac{x}{2})} d\tan(\tfrac{x}{2}) = \int \frac{1}{1+\tan(\frac{x}{2})} + \frac{1}{1-\tan(\frac{x}{2})} d\tan(\tfrac{x}{2})$$

$$= \ln\left|\frac{1+\tan(\frac{x}{2})}{1-\tan(\frac{x}{2})}\right| + C.$$

$$\frac{1+\tan(\frac{x}{2})}{1-\tan(\frac{x}{2})} = \frac{\cos(\frac{x}{2})+\sin(\frac{x}{2})}{\cos(\frac{x}{2})-\sin(\frac{x}{2})} = \frac{[\cos(\frac{x}{2})+\sin(\frac{x}{2})]^2}{\cos^2(\frac{x}{2})-\sin^2(\frac{x}{2})}$$

$$= \frac{1+\sin x}{\cos x}$$

$$\int \frac{1}{1+\cos x}\,dx = \int \frac{1}{1+\frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}}\,dx = \int \frac{1}{1+\frac{1-\tan^2(\frac{x}{2})}{1+\tan^2(\frac{x}{2})}}\, 2\,\frac{1}{\tan^2(\frac{x}{2})+1}\, d\tan(\tfrac{x}{2})$$

$$= \int \frac{1+\tan^2(\frac{x}{2})}{2}\, 2\, \frac{1}{\tan^2(\frac{x}{2})+1}\, d\tan(\tfrac{x}{2}) = \tan(\tfrac{x}{2}) + C$$

What is $I = \int \frac{1}{A\cos x + B\sin x + C}\,dx$? $t = \tan\frac{x}{2}$, we have $dx = \frac{2}{1+t^2}\,dt$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$.

$$I = \int \frac{1}{A(\frac{1-t^2}{1+t^2}) + B(\frac{2t}{1+t^2}) + C} \cdot \left(\frac{2}{1+t^2}\right) dt = \int \frac{2}{A(1-t^2) + 2Bt + C(1+t^2)}\,dt$$

$$= \int \frac{2}{(C-A)t^2 + (2B)t + (C+A)}\,dt$$

This goes back to what we have talked about for the rational function.

Sum to Product formula \ Product to sum formula

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right),$$

$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right),$$

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right),$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right),$$

$$\sin A\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\cos A\sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

$$\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\int \sin 5x\cos 3x\,dx = \frac{1}{2}\int[\sin 8x + \sin 2x]\,dx = \frac{1}{2}\left(\frac{-\cos 8x}{8} - \frac{\cos 2x}{2}\right) + C.$$

$$\int \cos x \cos^2 3x\,dx = \int \cos x\left[\tfrac{1}{2}(1+\cos 6x)\right]dx$$

$$= \frac{1}{2}\int \cos x\,dx + \frac{1}{2}\int \cos x \cos 6x\,dx$$

$$= \frac{1}{2}\int \cos x\,dx + \frac{1}{4}\int \cos 7x + \cos 5x\,dx$$

$$= \frac{\sin x}{2} + \frac{\sin 7x}{28} + \frac{\sin 5x}{10} + C.$$

$$\int \sin^m x \cos^n x\,dx$$

Case 1: m is odd

$$\int \sin^3 x \cos^2 x\,dx = -\int \sin^2 x \cos^2 x\,d\cos x = -\int (1-\cos^2 x)\cos^2 x\,d\cos x$$

$$= \int -\cos^2 x + \cos^4 x\,d\cos x = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + C.$$

Case 2: n is odd

$$\int \sin^4 x \cos^3 x\,dx = \int \sin^4 x \cos^2 x\,d\sin x = \int \sin^4 x(1-\sin^2 x)\,d\sin x$$

$$= \int \sin^4 x - \sin^6 x\,d\sin x = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C.$$

Case 3: m and n are even

$$\int \sin^2 x \cos^4 x\,dx = \int [\sin x \cos x]^2 \cos^2 x\,dx = \int \frac{1}{4}[\sin 2x]^2\,\frac{1+\cos 2x}{2}\,dx$$

$$= \frac{1}{16}\int [\sin 2x]^2\,(1+\cos 2x)\,d2x$$

$$= \frac{1}{16}\int (\sin 2x)^2\,d2x + \frac{1}{16}\int (\sin 2x)^2 \cos 2x\,d2x$$

$$= \frac{1}{16}\int 1-\cos 4x\,d(2x) + \frac{1}{16}\int (\sin 2x)^2\,d\sin 2x$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C$$

$\int \tan^m x \sec^n x dx$

Case 1: m is odd

$$\frac{\tan x}{\cos x} dx = d\frac{1}{\cos x} = \frac{\sin x}{[\cos x]^2}dx = \frac{\sin x}{\cos x}\frac{1}{\cos x}dx = \tan x \frac{1}{\cos x} dx.$$

$$\int \tan^3 x \frac{1}{\cos^4 x}dx = \int \tan^2 x \tan x \frac{1}{\cos^3 x}\frac{1}{\cos x}dx = \int \tan^2 x \frac{1}{\cos^3 x} d\frac{1}{\cos x}$$

$$= \int \frac{\sin^2 x}{\cos^2 x}\frac{1}{\cos^3 x} d\frac{1}{\cos x} = \int \left(\frac{1}{\cos^2 x} - 1\right)\frac{1}{\cos^3 x} d \frac{1}{\cos x} = \frac{1}{6} \frac{1}{\cos^6 x} - \frac{1}{4}\frac{1}{\cos^4 x} + C.$$

Case 2: n is even

$$\frac{1}{\cos^2 x}dx = d\tan x \quad \left(d\frac{\sin x}{\cos x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x}dx = \frac{1}{\cos^2 x}dx\right)$$

$$\int \tan^4 x \frac{1}{\cos^4 x}dx = \int \tan^4 x \frac{1}{\cos^2 x} d\tan x = \int \tan^4 x (1+\tan^2 x) d\tan x$$

$$= \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C.$$

Case 3: m is even and n is odd

$$\int \tan^4 \frac{1}{\cos^3 x}dx = \int \left(\frac{\sin^2 x}{\cos^2 x}\right)^2 \frac{1}{\cos^3 x}dx = \int \left(\frac{1}{\cos^2 x}-1\right)^2 \frac{1}{\cos^3 x}dx$$

$$= \int \frac{1}{\cos^7 x} - \frac{2}{\cos^5 x} + \frac{1}{\cos^3 x} dx$$

Actually, we have the iteration formula

$$\int \left(\frac{1}{\cos x}\right)^n dx = \frac{1}{n-1}\left(\frac{1}{\cos x}\right)^{n-2}\tan x + \frac{n-2}{n-1}\int \left(\frac{1}{\cos x}\right)^{n-2}dx.$$

The final solution is

$$\frac{1}{6}\left(\frac{1}{\cos x}\right)^5 \tan x - \frac{7}{24}\left(\frac{1}{\cos x}\right)^3 \tan x + \frac{1}{16}\left(\frac{1}{\cos x}\right)\tan x + \frac{1}{16}\ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + C.$$

Integration of irrational Functions:
(Use polar coordinates)

① $\int x^3\sqrt{4-x^2}\,dx$

we let $y=\sqrt{4-x^2} \Rightarrow x^2+y^2=4$. This means we can use polar decomposition.

$x=2\sin\theta$

$$\int r^3\sqrt{4-x^2}\,dx = \int 8\sin^3\theta\sqrt{4\cos^2\theta}\,(2\cos\theta)d\theta = 32\int \sin^3\theta\cos^2\theta\,d\theta$$

This goes back to the case $\int \sin^n x\cos^m x\,dx$. We have

$$32\int\cos^2\theta\sin^3\theta\,d\theta = 32\int\cos^2\theta(1-\cos^2\theta)\,d(-\cos\theta) = 32\int\cos^4\theta - \cos^2\theta\,d\cos\theta$$

$$= \frac{32}{5}\cos^5\theta - \frac{32}{3}\cos^3\theta + C.$$

$$= \frac{32}{5}\left(\frac{\sqrt{4-x^2}}{2}\right)^5 - \frac{32}{3}\left(\frac{\sqrt{4-x^2}}{2}\right)^3 + C$$

② $\int\frac{\sqrt{x^2-4}}{x^3}dx$ (In this case, we should use another decomposition)

$$x=\frac{2}{\cos\theta},\quad dx=\frac{2\sin\theta}{\cos^2\theta}d\theta = 2\tan x\frac{1}{\cos x}d\theta.$$

$$\Rightarrow\int\frac{\sqrt{x^2-4}}{x^3}dx = \int\frac{\sqrt{4+\frac{4}{\cos^2\theta}}}{8\frac{1}{\cos^3\theta}}dx = \int\frac{\sqrt{4+\frac{4}{\cos^2\theta}}}{\frac{8}{\cos^3\theta}}\,2\frac{\sin\theta}{\cos^2 x}d\theta$$

$$=\int\frac{\sin\theta\sqrt{4+\frac{4}{\cos^2\theta}}}{\frac{4}{\cos\theta}}d\theta = \int\frac{\sin\theta\sqrt{4\cos^2\theta+4}}{4}d\theta = \frac{1}{2}\int\sin^2\theta\,d\theta$$

$$=\frac{1}{4}\int 1-\cos 2\theta\,d\theta = -\frac{1}{8}\sin 2\theta + \frac{\theta}{4} + C$$

$$\cos\theta=\frac{2}{x},\quad \sin\theta=\sqrt{1-\frac{4}{x^2}} = \frac{\sqrt{x^2-4}}{x}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = \frac{4\sqrt{x^2-4}}{x^2} \Rightarrow -\frac{1}{8}\sin 2\theta = -\frac{\sqrt{x^2-4}}{2x^2}$$

$$\theta = \cos^{-1}\frac{2}{x}$$

$$\Rightarrow -\frac{\sqrt{x^2-4}}{2x^2} + \frac{1}{4}\cos^{-1}\frac{2}{x} + C$$

Exercise

For $a>0$, we have

a) $\int \sqrt{a^2-x^2}\,dx = ?$

we let $x = a\sin\theta$ ($|x|\le a$),

$$\int \sqrt{a^2-a^2\sin^2\theta}\,dx = a^2\int \cos\theta\, d\sin\theta = a^2\int \cos^2\theta\, d\theta$$

$$= a^2\int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{a^2}{4}\int 1+\cos 2\theta\, d2\theta = \frac{a^2}{4}(2\theta+\sin 2\theta)$$

$\theta = \arcsin\frac{x}{a}$, $\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{x}{a}\sqrt{1-\sin^2\theta} = \frac{2x}{a}\sqrt{1-\frac{x^2}{a^2}}$.

we have

$$\int \sqrt{a^2-a^2\sin^2\theta}\,dx = \frac{a^2}{4}\left(2\arcsin\frac{x}{a} + \frac{2x}{a}\sqrt{1-\frac{x^2}{a^2}}\right) = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$$

b) $\int \sqrt{x^2-a^2}\,dx = ?$

$x = a\frac{1}{\cos\theta}$, $x^2-a^2 = a^2\frac{1}{\cos^2\theta} - a^2 = a^2\frac{1-\cos^2\theta}{\cos^2\theta} = a^2\frac{\sin^2\theta}{\cos^2\theta}$.

$$dx = a\,d\frac{1}{\cos\theta} = a\frac{\sin\theta}{\cos^2\theta}\,d\theta = \frac{a}{\cos\theta}\tan\theta\, d\theta$$

$$\int \sqrt{x^2-a^2}\,dx = \int a\left|\frac{\sin\theta}{\cos\theta}\right|\frac{a}{\cos\theta}\tan\theta\,d\theta = a^2\int \left|\frac{\sin\theta}{\cos\theta}\right|\frac{1}{\cos\theta}\tan\theta\,d\theta$$

$$= a^2\int \left|\frac{\sin\theta}{\cos\theta}\right|\frac{\sin\theta}{\cos^2\theta}\,d\theta.$$

If $\theta\in[0,\frac{\pi}{2})$, we have $\int \sqrt{x^2-a^2}\,dx = a^2\int \tan^2\theta\frac{1}{\cos\theta}\,d\theta = a^2\int \frac{1-\cos^2\theta}{\cos^2\theta}\frac{1}{\cos\theta}\,d\theta$

$$= a^2\int \frac{1}{\cos^3\theta} - \frac{1}{\cos\theta}\,d\theta$$

If $\theta\in(\frac{\pi}{2},\pi]$, $\int \sqrt{x^2-a^2} = a^2\int -\frac{\sin^2\theta}{\cos^2\theta}\frac{1}{\cos\theta}\,d\theta \Rightarrow -a^2\int \frac{1}{\cos^3\theta} - \frac{1}{\cos\theta}\,d\theta$

$\theta\in[0,\frac{\pi}{2})$,

$$\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + C$$

$\theta\in(\frac{\pi}{2},\pi]$ $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln\left|x-\sqrt{x^2-a^2}\right| + C$

Note that

$$(x+\sqrt{x^2-a^2})(x-\sqrt{x^2-a^2})=a^2$$

$$\Rightarrow \int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln\left|\frac{a^2}{x+\sqrt{x^2-a^2}}\right| + C$$

$$= \frac{x}{2}\sqrt{x^2-a^2} + \frac{a^2}{2}\ln a^2 - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C$$

$$= \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln|x+\sqrt{x^2-a^2}| + C.$$

More things, $\int \frac{1}{\cos^3 x}dx$

$$\int \frac{1}{\cos^3 x}dx = \frac{1}{\cos x}\frac{\sin x}{\cos x} - \int \frac{\sin x}{\cos x}\frac{1}{\cos x}\frac{\sin x}{\cos x}dx$$

$$d\frac{\sin x}{\cos x} = \frac{\cos^2 x+\sin^2 x}{\cos^2 x}dx = \frac{1}{\cos^2 x}dx$$

$$\int \frac{1}{\cos^3 x}dx = \int \frac{1}{\cos x}\,d\frac{\sin x}{\cos x} = \frac{1}{\cos x}\frac{\sin x}{\cos x} - \int \frac{\sin x}{\cos x}\,d\frac{1}{\cos x}$$

$$= \frac{1}{\cos x}\frac{\sin x}{\cos x} - \int \frac{\sin x}{\cos x}\frac{\sin x}{\cos^2 x}dx$$

$$= \frac{\sin x}{\cos^2 x} - \int \frac{1-\cos^2 x}{\cos^3 x}dx = \frac{\sin x}{\cos^2 x} - \int \frac{1}{\cos^3 x}dx + \int \frac{1}{\cos x}dx$$

$$\Rightarrow 2\int \frac{1}{\cos^3 x}dx = \frac{\sin x}{\cos^2 x} + \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right| + C.$$

Question

① $\int \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x) + C$

② $\int \frac{1}{\sqrt{1+x^2}}\, dx = ?$

$x = \tan\theta \quad dx = \frac{d\tan\theta}{d\theta}\, d\theta \Rightarrow \frac{d\frac{\sin\theta}{\cos\theta}}{d\theta}\, d\theta = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\, d\theta = \frac{1}{\cos^2\theta}\, d\theta$

$\Rightarrow \int \frac{1}{\sqrt{1+x^2}}\, dx = \int \frac{1}{\sqrt{1+\tan^2\theta}}\, dx = \int \frac{1}{\sqrt{1+\tan^2\theta}} \frac{1}{\cos^2\theta}\, d\theta$

$= \int \cos\theta \frac{1}{\cos^2\theta}\, d\theta = \int \frac{1}{\cos\theta}\, d\theta$ (t-substitution)

$= \ln\left(\frac{1}{\cos\theta} + \tan\theta\right) + C$

$= \ln(x + \sqrt{x^2+1}) + C$

③ $\int \frac{1}{\sqrt{A+Bx^2}} = (B>0) \quad A>0$

$= \frac{1}{\sqrt{B}} \int \frac{1}{\sqrt{\frac{A}{B}+x^2}} \Rightarrow \boxed{\ln|x + \sqrt{a^2+x^2}| + C} = \int \frac{1}{\sqrt{a^2+x^2}}\, dx$

$\Downarrow$

$\frac{1}{\sqrt{B}} \ln|x + \sqrt{(\frac{A}{B})^2 + x^2}|$

④ $\begin{cases} B<0 \\ A>0 \end{cases} \quad \int \frac{1}{\sqrt{A+Bx^2}} = \frac{1}{\sqrt{-B}} \arcsin\left(x\sqrt{\frac{-B}{A}}\right)$

$\Leftarrow \int \frac{1}{\sqrt{a^2-x^2}}\, dx = \arcsin\left(\frac{x}{a}\right)$

$+C$

(5) What is Ax^2+Bx+C

use 'δ' to disscuss!

(1) $A<0$, $\delta>0$, $\int \frac{1}{\sqrt{Ax^2+Bx+C}}dx = -\frac{1}{\sqrt{-A}} \arcsin\left(\frac{2Ax+B}{\sqrt{B^2-4AC}}\right)+C$

(2) $A>0$, $Ax^2+Bx+C = A(x+\frac{B}{2A})^2 + A(\frac{C}{A}-\frac{B^2}{4A^2})$

$u=\sqrt{A}(x+\frac{B}{2A})$, $k=\sqrt{A}\sqrt{\frac{C}{A}-\frac{B^2}{4A^2}}$

It is easy to know $\int \frac{1}{\sqrt{u^2+k^2}}du$!

(6) $\int \frac{1}{\sqrt{Ax^3+Bx^2+Cx+D}}$ X No (Elliptic Integral)

Exercise $\int x^m (\ln x)^n dx$? m and $n>1$, integers

$I=\int x^m (\ln x)^n dx$, $u=\ln x$

$I=\int e^{mu}\cdot u^n \cdot e^u du = \int u^n e^{(m+1)u} du$

$J(n)=\int u^n e^{au} du$, $a=m+1$

$= \frac{u^n e^{au}}{a} - \frac{n}{a}\int u^{n-1}e^{au}du = \frac{u^n e^{au}}{a} - \frac{n}{a}J(n-1)$

$\Rightarrow J(n) = \frac{ue^{au}}{a} - \cdots$

$\vdots$

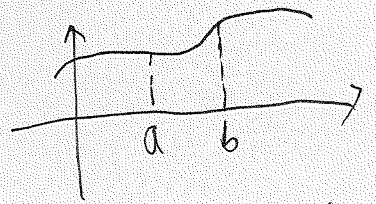
$J(n) = e^{au}\sum_{j=0}^{n}(-1)^j \frac{n!}{(n-j)!}\cdot\frac{u^{n-j}}{a^{j+1}} + C$

Exercise ① $\int x^{-m}(\ln x)^n dx$ Some can be solved! Try it

② $\int x^{-m}(\ln x)^{-n} dx$

③ $\int x^m (\ln x)^{-n} dx$

Definite Integration.



We splite $[a,b]$ into many parts, $P_k = \{ \underset{x_0}{a} < x_1 < x_2 < \cdots < x_k = b \}$.

In $[x_0, x_1]$, we let $m_0 = \inf\{f(x) : x_0 \le x \le x_1\}$, we let $M_0 = \sup\{f(x), x_0 \le x \le x_1\}$.
we have

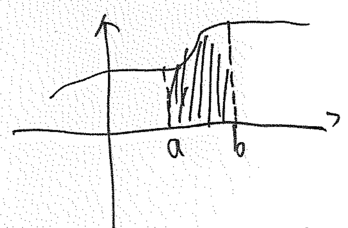
$$f(x) = \sum_{h=0}^{k-1} m_h \Delta x_h \quad \text{or} \quad f(x) = \sum_{h=0}^{k-1} M_h \Delta x_h .$$

We call this thing the Riemann sum, If upper sum or the lower sum are equal. it is Riemann integrable as $k \to \infty$.

$\int_a^b f(x) = F(x)\Big|_a^b = F(b) - F(a)$, in most cases, it is similar to indefinite integral. But there are still many cases we need to be carefull!

$$\int_0^1 x^2 = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$$

Geometry meaning:



$\int_a^b f(x)dx$ means the area of the shawdowed part.

Some things we need to care about

(1) The interval we consider is related to "x"!

Consider this type of integration, what is

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(y) = ?$$

Leibniz rule for

$$\frac{d}{dx}\int_{a(x)}^{b(x)} F(y)\,dy = F(b(x))\, b'(x) - F(a(x))\, a'(x).$$

Let us consider some examples:

$$F(x)=\int_x^{x^2} e^{\cos t}\,dt$$

$$\frac{dF}{dx}=\frac{d}{dx}\int_x^{x^2} e^{\cos t}\,dt = e^{\cos x^2}(2x) - e^{\cos x}(1)$$
$$= 2x\,e^{\cos x^2} - e^{\cos x}.$$

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we can also use fundamental theorem of calculus to evaluate limit of series.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$
$$=\lim_{n\to\infty}\frac{1}{n}\left(f\left(\tfrac{1}{n}\right)+f\left(\tfrac{2}{n}\right)+f\left(\tfrac{3}{n}\right)+\cdots+f\left(\tfrac{n}{n}\right)\right)$$
$$=\int_0^1 f(x)\,dx.$$

Here we use the definition of the Riemann sum. Find

① $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}$

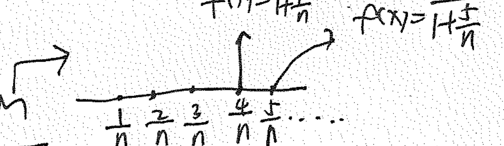
② $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}$

③ $\displaystyle\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{\sqrt{n+k}}$

Answer: $\displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{n}{n+k}=\lim_{n\to+\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\frac{k}{n}}$

$$=\int_0^1\frac{1}{1+x}\,dx=[\ln(1+x)]_0^1=\ln 2.$$

Here we make more explanations.



$$\int_0^1\frac{1}{1+x}=\sum_{k=1}^{n}\underbrace{\frac{1}{n}}\ \frac{1}{1+\frac{k}{n}}$$

$\frac{1}{1+\frac{k}{n}}$ represents the value of $f(x)$

we split $[0,1]$ into n parts, $\frac{1}{n}$ is the length of one part.

2 $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{n^2}{n^2+k^2} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+(\frac{k}{n})^2} = \int_0^1 \frac{1}{1+x^2}dx = [\tan^{-1}x]_0^1$

$= \frac{\pi}{4}$

3. $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\sum_{n=1}^{n}\frac{1}{\sqrt{n+k}} = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{1+\frac{k}{n}}} = \int_0^1\frac{1}{\sqrt{1+x}}dx = [2\sqrt{1+x}]|_0^1$

$= 2(\sqrt{2}-1).$

Find $\lim_{n\to\infty}\frac{\sqrt[n]{(n+1)(n+2)\cdots\cdots(2n)}}{n}$

Answer $= \ln\left(\lim_{n\to\infty}\frac{\sqrt[n]{(n+1)(n+2)\cdots(2n)}}{n}\right)$

$= \lim_{n\to\infty}\ln\frac{\sqrt[n]{(n+1)\cdots(2n)}}{n}$

$= \lim_{n\to\infty}\frac{1}{n}\ln((1+\frac{1}{n})\cdots\cdots(1+\frac{n}{n}))$

$= \lim_{n\to\infty}\frac{1}{n}(\ln(1+\frac{1}{n})+\ln(1+\frac{2}{n})+\cdots\cdots+\ln(1+\frac{n}{n}))$

$= \int_0^1\ln(1+x)dx = [(x+1)\ln(1+x)-x]_0^1 = 2\ln 2-1.$

In tegration using substitution

We have $\int x^3dx = \frac{1}{2}\int x^2dx^2 = \frac{1}{2}\frac{1}{2}(x^2)^2 = \frac{1}{4}x^4$. Here we use substitution $t=x^2$. For the definite integral, the idea is similar. but there are still something we need to be care about.

1. Evaluate

$\int_0^1 8x(x^2+1)dx \underset{u=x^2+1}{\Rightarrow} \int_1^2 8u\frac{1}{2}du \Rightarrow \int_1^2 4u\,du = [2u^2]_1^2 = 6.$

The range will change if we use integration by substitution

Evaluate $\int_e^{e^2} \frac{1}{x\ln x}dx$

Let $u=\ln x$, $du=\frac{1}{x}dx$

$$\int_e^{e^2}\frac{1}{x\ln x}dx = \int_1^2 \frac{1}{u}du = [\ln u]_1^2 = \ln 2$$

What is the integration by parts for definite integral?

$$\int_a^b u\frac{dv}{dx}dx = [uv]\Big|_a^b - \int_a^b v\frac{du}{dx}dx$$

Evaluate

$$\int_1^e x\ln x\,dx = \int_1^e \ln x \frac{d\frac{x^2}{2}}{dx}dx = \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}\frac{d\ln x}{dx}dx$$

$$= \left(\frac{e^2}{2}\ln e - \frac{1}{2}\ln 1\right) - \int_1^e \frac{x}{2}dx$$

$$= \frac{e^2}{2} - \left[\frac{x^2}{4}\right]_1^e = \frac{e^2}{2} - \left(\frac{e^2}{4}-\frac{1}{4}\right) = \frac{e^2}{4}+\frac{1}{4}.$$

How to think? (Just do indefinite intergral, substitute the result later)

Improper integrals

In most cases, we only consider integration on bounded domain and $f(x)$ is bounded. However, we have to consider $f(x)\to+\infty$ and unbounded domain sometimes.

1. $\int_0^{+\infty}\frac{1}{(x+1)(3x+2)}$

$$\int_0^L \frac{1}{(x+1)(3x+2)} = \left[-\ln|x+1| + \ln|3x+2|\right]\Big|_0^L,$$ consider the limit

$$\lim_{L\to+\infty}\int_0^L \frac{1}{(x+1)(3x+2)} = \ln 3 - \ln 2.$$

Find $\int_0^{+\infty} x e^{-2x} dx$

Answer: Consider $\lim_{L \to +\infty} \int_0^L x e^{-2x} dx$

$$\int_0^L xe^{-2x} dx = \left[-\frac{1}{2} xe^{-2x}\right]_0^L + \left[-\frac{1}{4} e^{-2x}\right]_0^L$$

$$\Rightarrow \lim_{L \to +\infty} \int_0^L xe^{-2x} dx = \frac{1}{4}$$

In two cases above, we only consider the domain is unbounded. Now, we consider function will goes to the infinity in some intervals.

Find $\int_0^1 \frac{1}{\sqrt{1-x}} dx$

$\frac{1}{\sqrt{1-x}} = \infty$ at $x=1$!

Answer: $\lim_{L \to 1^-} \int_0^L \frac{1}{\sqrt{1-x}} dx = \lim_{L \to 1^-} \left[-2\sqrt{1-x}\right]_0^L = \lim_{L \to 1^-} -2\sqrt{1-L} + 2 = 2$.

Find: $\int_0^{\infty} \frac{1}{(1+x)\sqrt{x}} dx$

(1) The domain is unbounded

(2) at $x=0$, $\frac{1}{(1+x)\sqrt{x}} = \infty$!

$x = u^2$, $dx = 2u\,du$

$$\int_0^\infty \frac{1}{(1+u^2)\cdot u} 2u\,du = \int_0^\infty \frac{2}{1+u^2}\,du = 2[\arctan(u)]_0^\infty$$

$$= \pi.$$