MMAT5320 Computational Mathematics

Academic year 2025/26
Practice Problems Set 1

1. Let $A \in \mathbb{C}^{m\times m}$ and unitary matrix. Let $A = U\Sigma V^*$ be the SVD decomposition of A, where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{m\times m}$ are unitary, and $\Sigma \in \mathbb{C}^{m\times m}$ has the singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_m \geq 0$ in its diagonal.

   (a) Prove that if A is hermitian, then all eigenvalues of A are 1 or -1.

   (b) Prove that $\sigma_1 = \cdots = \sigma_m = 1$.

   Solution.

   (a) Assume $A = A^*$ and let λ be eigenvalue of A and x be the corresponding eigenvector. Then

$$\begin{aligned} Ax &= \lambda x \\ A^*Ax &= A^*\lambda x \\ x &= \lambda A^* x \\ x &= \lambda Ax \\ x &= \lambda^2 x \end{aligned}$$

   Hence, $\lambda = 1$ or -1.

   (b) First note that $\|A\|_2 = \sigma_1$ and A is unitary, then $\sigma_1 = \|A\|_2 = 1$.
   Next,

$$1 = \|A\|_2 = \|A^*\|_2 = \|A^{-1}\|_2 = \|\Sigma^{-1}\|_2 = \sigma_m$$

   Since $\sigma_1 = \max_j \sigma_j$ and $\sigma_m = \min_j \sigma_j$, then $\sigma_1 = \cdots = \sigma_m = 1$.

□

2. Let $A \in \mathbb{C}^{m\times m}$ be hermitian. Let $A = U\Sigma V^*$ be the SVD decomposition of A.

   (a) Prove that all eigenvalues of A are real.

   (b) Prove that the square of singular values of A is the square of eigenvalues of A.

   Solution.

   (a) For each eigenvalue λ, we have $x^*Ax = x^*\lambda x = \lambda\|x\|^2$, then by taking conjugate on both side

$$\lambda\|x\|^2 = x^*Ax = (x^*Ax)^* = (\lambda\|x\|^2)^* = \lambda^*\|x\|^2$$

   Hence, $\lambda = \lambda^*$, that is λ is real.

   (b) Note that

$$A^2 = A^*A = (U\Sigma V^*)^*(U\Sigma V^*) = V\Sigma^2 V^*$$

   The eigenvalues of A^2 are the eigenvalue of Σ^2, that is $\{\sigma_1^2, \cdots, \sigma_m^2\}$.
   Also, if λ and x are the eigenvalue and eigenvector of A, $A^2x = \lambda Ax = \lambda^2 x$, thus λ^2 is the eigenvalue of A^2. Hence, the square of singular values of A is the square of eigenvalues of A.

□

3. Vector and matrix p-norm are related by various inequalities, often involving the dimensions m or n. For each of the following, verify the inequality and give an example of a nonzero vector or matrix (for general m, n) for which equality is achieved. In this problem x is an m-vector and A is an $m \times n$ matrix.

   (a) $\|x\|_1 \leq \sqrt{m}\|x\|_2$

   (b) $\|x\|_2 \leq \|x\|_1$

   (c) $\|A\|_1 \leq \sqrt{m}\|A\|_2$

   (d) $\|A\|_2 \leq \sqrt{n}\|A\|_1$

   Solution.

   (a) $\|x\|_1^2 = (\sum_{i=1}^{m} |x_i|)^2 \leq m \sum_{i=1}^{m} |x_i|^2 = m\|x\|_2^2$
   example: $x = (0, \cdots, 0, 1, 0, \cdots, 0)$.

   (b) $\|x\|_2^2 = \sum_{i=1}^{m} |x_i|^2 \leq (\sum_{i=1}^{m} |x_i|)^2 = \|x\|_1^2$
   example: $x = (1, \cdots, 1)$.

   (c) $\|A\|_1 = \sup_{x \neq 0, \|x\|_1 \leq 1} \|Ax\|_1 \leq \sup_{x \neq 0, \|x\|_1 \leq 1} \sqrt{m}\|Ax\|_2 \leq \sup_{x \neq 0, \|x\|_2 \leq 1} \sqrt{m}\|Ax\|_2 \leq \sqrt{m}\|A\|_2$
   example:

$$A = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & \\ 1 & 0 & \cdots & 0 \end{bmatrix}.$$

   (d) $\|A\|_2 = \sup_{x \neq 0, \|x\|_2 \leq 1} \|Ax\|_2 \leq \sup_{x \neq 0, \|x\|_2 \leq 1} \|Ax\|_1 \leq \sup_{x \neq 0, \|x\|_1 \leq 1} \sqrt{n}\|Ax\|_1 = \sqrt{n}\|A\|_1$ example:

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

□

4. Prove that the unique orthogonal projection from $\mathbb{C}^n$ to $\mathbb{C}^n$, which is unitary and identity.

   Solution.
   Let P be the orthogonal projection which is unitary. Then we have $P^2 = P$, $P = P^*$ and $PP^* = P^*P = I$. Thus,

$$P = P^2 = PP^* = I$$

□

5. If A is skew hermitian, i.e. $A^* = -A$, prove that $(I - A)^{-1}(I + A)$ is unitary.

   Solution.
   First note that $I^2 - A^2 = (I - A)(I + A) = (I + A)(I - A)$, then we multiply both side by $(I - A)^{-1}$ pre and post,

$$(I + A)(I - A)^{-1} = (I - A)^{-1}(I + A)$$

   Similarly, we have

$$(I + A^*)(I - A^*)^{-1} = (I - A^*)^{-1}(I + A^*)$$

Then we have

$$\begin{aligned}(I-A)^{-1}(I+A)((I-A)^{-1}(I+A))^* &= (I-A)^{-1}(I+A)(I+A^*)(I-A^*)^{-1}\\ &= (I+A)(I-A)^{-1}(I-A)(I+A)^{-1}\\ &= (I+A)(I+A)^{-1}\\ &= I\end{aligned}$$

On the other hand,

$$\begin{aligned}((I-A)^{-1}(I+A))^*(I-A)^{-1}(I+A) &= (I+A^*)(I-A^*)^{-1}(I-A)^{-1}(I+A)\\ &= (I-A)(I+A)^{-1}(I+A)(I-A)^{-1}\\ &= (I-A)(I-A)^{-1}\\ &= I\end{aligned}$$

□

6. Let $\mathbb{C}^m = S_1 + S_2$ and let $v_1, \cdots, v_r$ be a basis of S_1, $v_{r+1}, \cdots, v_m$ be a basis of S_2 and $V = [v_1|\cdots|v_m]$. Show that

$$P = V \cdot \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \cdot V^{-1}$$

is the projector onto S_1 along S_2.

Solution.
First, we want to show P is projection. Let $x \in \mathbb{C}^m$ and we have $x = \sum\limits_{i=1}^{m} a_i v_i$.

$$Px = V \cdot \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \cdot V^{-1}x = V \cdot \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = V \begin{bmatrix} a_1 \\ \vdots \\ a_r \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \sum_{i=1}^{r} a_i v_i$$

Then, we have $P^2x = P\sum\limits_{i=1}^{r} a_i v_i = \sum\limits_{i=1}^{r} a_i v_i$=Px. Hence, P is projection.
Next, Since $Px = \sum\limits_{i=1}^{r} a_i v_i$, range$(P) = S_1$.
Then, let $x \in$ null(P) and $x = \sum\limits_{i=1}^{m} a_i v_i$, thus

$$\sum_{i=1}^{r} a_i v_i = Px = 0$$

So, we have $a_i = 0$ if $i \leq r$. Hence, null$(P) = S_2$. To conclude, P is the projector onto S_1 along S_2. □

7. (a) Find the orthogonal projector P_C onto range(A), where A is the following matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 4 \end{bmatrix}$$

(b) Find the orthogonal projector P_R onto range(A^*).

(c) What is $P_C A P_R$? Prove your result algebraically.

Solution.

(a) Let $A = [a_1|a_2|a_3]$. Since $a_1 = 2a_2 = a_3$,

$$P_C = \frac{a_1 a_1^*}{\|a_1\|_2^2} = \frac{1}{17}\begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}$$

(b) Let $A^* = [b_1|b_2]$.Since $b_1 = 4b_2$,

$$P_R = \frac{b_1 b_1^*}{\|b_1\|_2^2} = \frac{1}{6}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

(c)

$$\begin{aligned} P_C A P_R &= \frac{1}{17}\begin{bmatrix} 1 & 4 \\ 4 & 16 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 4 \end{bmatrix}\frac{1}{6}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 1 \\ 4 & 8 & 4 \end{bmatrix} \\ &= A \end{aligned}$$

Since P_C is orthogonal projector onto range(A), we have for any vector v, $P_C A v = Av$, that is $P_C A = A$. Next, Since P_R is orthogonal projector onto range(A^*), we have for any vector w and v,

$$(AP_R v)^* w = (AP_R^* v)^* w = v^* P_R A^* w = v^* A^* w = (Av)^* w.$$

Thus, $AP_R = A$. Hence, $P_C A P_R = P_C A = A$.

□

8. Consider the matrices

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}$$

What is the orthogonal projector P onto range(A), and what is the image under P of the vector $(1, 2, 3)^*$?

Solution.

$$\begin{aligned} P &= A(A^*A)^{-1}A^* \\ &= \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 2 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \end{aligned}$$

$$P\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$$

□

9. Let $P \in \mathbb{C}^{m\times m}$ be a nonzero orthogonal projector and $\text{null}(P) \neq 0$. Show that only 0 and 1 are eigenvalues of P.

Solution.
Let W_1 be the range of P and W_2 be the null space of P. Since P is orthogonal projection, $\mathbb{C}^m = W_1 + W_2$ and W_1 and W_2 are orthogonal. Let $\{v_1, \cdots, v_r\}$ be the set of basis of W_1 and $\{v_{r+1}, \cdots, v_m\}$ be the set of basis of W_2. Then we have

$$\begin{aligned} Pv_i &= 1 \times v_i, \quad \forall i \le r \\ Pv_i &= 0 \times v_i, \quad \forall i > r \end{aligned}$$

Hence, v_i's are the eigenvectors of P and 0 and 1 are the eigenvalues of P. Assume λ is eigenvalue of P which is not 0 or 1 and v is the corresponding eigenvector. Since $\mathbb{C}^m = W_1 + W_2$, $v = \sum_i a_i v_i$, then

$$\lambda v = Pv = P\sum_i a_i v_i = \sum_{i\le r} a_i v_i$$

which is contradiction. Thus, only 0 and 1 are eigenvalues of P. □

10. Consider the matrices A and B

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

(a) Using Gram-Schmidt iteration, determine a reduced QR factorization $A = \hat{Q}\hat{R}$ and a full QR factorization $A = QR$.

(b) Again using any method you like, determinea reduced $B = \hat{Q}\hat{R}$ and full QR factorization $B = QR$.

Solution.

(a) Let a_1 and a_2 be the first and second column of A respectively. Then

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad r_{11} = \|a_1\|_2 = 2\sqrt{2}$$

Then

$$r_{12} = q_1^* a_2 = \frac{1}{\sqrt{2}}, \quad q_2 = \frac{a_2 - r_{12}q_1}{\|a_2 - r_{12}q_1\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad r_{22} = \|a_2 - r_{12}q_1\|_2 = \frac{\sqrt{3}}{\sqrt{2}}$$

Hence,

$$A = \hat{Q}\hat{R} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} 2\sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} \end{bmatrix}$$

Next we need to find q_3 such that $q_1^* q_3 = 0$, $q_2^* q_3 = 0$ and $\|q_3\|_2$. Then $q_3 = [\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0]$,

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \end{bmatrix}\begin{bmatrix} 2\sqrt{2} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}$$

(b) Let b_1 and b_2 be the first and second column of B respectively. Then

$$q_1 = \frac{b_1}{\|b_1\|} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad r_{11} = \|b_1\|_2 = 3$$

Then

$$r_{12} = q_1^* b_2 = 1, \quad q_2 = \frac{b_2 - r_{12}q_1}{\|b_2 - r_{12}q_1\|} = \frac{1}{3}\begin{bmatrix}1\\2\\-2\end{bmatrix}, \quad r_{22} = \|b_2 - r_{12}q_1\|_2 = 1$$

Hence,

$$B = \hat{Q}\hat{R} = \begin{bmatrix}\frac{2}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3}\end{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 1\end{bmatrix}$$

Next we need to find q_3 such that $q_1^* q_3 = 0$, $q_2^* q_3 = 0$ and $\|q_3\|_2$. Then $q_3 = [-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}]$,

$$B = QR = \begin{bmatrix}\frac{2}{3} & \frac{1}{3} & \frac{2}{3}\\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3}\\ \frac{2}{3} & -\frac{2}{3} & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}3 & 1\\ 0 & 1\\ 0 & 0\end{bmatrix}$$

□

11. Determine the (a) eigenvalues, and (b) singular values of a Householder reflector

$$F = I - 2\frac{vv^*}{v^*v}.$$

Solution.

(a) Let λ and x be the eigenvalue and eigenvector of F. Then

$$\begin{aligned} Fx &= \lambda x \\ (I - \frac{vv^*}{v^*v})x - \frac{vv^*}{v^*v}x &= \lambda x \end{aligned}$$

Consider the first case $x = cv$, $c \neq 0$,

$$\begin{aligned} (I - \frac{vv^*}{v^*v})x - \frac{vv^*}{v^*v}x &= \lambda x \\ 0 - \frac{vv^*}{v^*v}x &= \lambda x \\ -x &= \lambda x \end{aligned}$$

Thus, $\lambda = -1$.
Next, consider the second case $x \perp v$,

$$\begin{aligned} (I - \frac{vv^*}{v^*v})x - \frac{vv^*}{v^*v}x &= \lambda x \\ (I - \frac{vv^*}{v^*v})x - 0 &= \lambda x \\ x &= \lambda x \end{aligned}$$

Thus, $\lambda = 1$.
Finally, consider the third case $x = x_1 + x_2$, where $x_1 = cv$ and $x_2 \perp v$,

$$\begin{aligned} (I - \frac{vv^*}{v^*v})x - \frac{vv^*}{v^*v}x &= \lambda x \\ x_2 - x_1 &= \lambda x \end{aligned}$$

Thus, x is not eigenvector. Hence, eigenvalues of F are 1 and -1.

(b) The singular values will be the positive square roots of the eigenvalues of F^*F,

$$\begin{aligned} F^*F &= \left(I - 2\frac{vv^*}{v^*v}\right)^* \left(I - 2\frac{vv^*}{v^*v}\right) \\ &= \left(I - 2\frac{vv^*}{v^*v}\right) \left(I - 2\frac{vv^*}{v^*v}\right) \\ &= I - 4\frac{vv^*}{v^*v} + 4\frac{vv^*}{v^*v}\frac{vv^*}{v^*v} \\ &= I - 4\frac{vv^*}{v^*v} + 4\frac{vv^*}{v^*v} \\ &= I \end{aligned}$$

Thus, the eigenvalue of F^*F is 1 and hence, all of the singular values will be 1.

□

12. Suppose the $m \times n$ matrix A has the form

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

where A_1, is a nonsingular matrix of dimension $n \times n$ and A_2 is an arbitrary matrix of dimension $(m-n) \times n$. Prove that $\|A^+\|_2 \leq \|A_1^{-1}\|_2$.

Solution.
Let $P \in \mathbb{C}^{m \times m}$ be the orthogonal projector onto range(A). For any $x \in \mathbb{C}^m$, we have $x = Px + (x - Px)$, where $Px \in$ range(A) and $x - Px \in$ null(A). Thus, we have

$$\begin{aligned} (x - Px)^* Ay = 0, &\quad \forall y \in \mathbb{C}^m \\ y^* A^*(x - Px) = 0, &\quad \forall y \in \mathbb{C}^m \end{aligned}$$

Hence, $A^*(x - Px) = 0$.

$$\begin{aligned} \|A^+\|_2 &= \sup_{x \neq 0} \frac{\|A^+ x\|_2}{\|x\|_2} \\ &= \sup_{x \neq 0} \frac{\|(A^*A)^{-1}A^*(Px + x - Px)\|_2}{\|Px + x - Px\|_2} \\ &\leq \sup_{x \neq 0} \frac{\|(A^*A)^{-1}A^*Px\|_2}{\|Px\|_2} \\ &\leq \sup_{y \neq 0} \frac{\|(A^*A)^{-1}A^*Ay\|_2}{\|Ay\|_2} \\ &\leq \sup_{y \neq 0} \frac{\|y\|_2}{\|Ay\|_2} \\ &\leq \sup_{y \neq 0} \frac{\|y\|_2}{\sqrt{\|A_1 y\|_2^2 + \|A_2 y\|_2^2}} \\ &\leq \sup_{y \neq 0} \frac{\|y\|_2}{\|A_1 y\|_2} \\ &\leq \sup_{y \neq 0} \frac{\|A_1^{-1} y\|_2}{\|y\|_2} \\ &= \|A_1^{-1}\|_2 \end{aligned}$$

□

13. Solve the least square problem: find $x \in \mathbb{C}^3$ such that $\|b - Ax\|_2$ is minimized by (a) normal equation and (b) QR factorization, where

$$A = \begin{bmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix}$$

Solution.

(a)

$$A^*A = \begin{bmatrix} 3 & 4 & 0 \\ -6 & -8 & 1 \end{bmatrix} \begin{bmatrix} 3 & -6 \\ 4 & -8 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 25 & -50 \\ -50 & 101 \end{bmatrix}$$

and

$$A^*b = \begin{bmatrix} 3 & 4 & 0 \\ -6 & -8 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 25 \\ -48 \end{bmatrix}$$

Then by solving

$$A^*Ax = A^*b,$$

we have

$$x = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

(b) $A = \hat{Q}\hat{R}$ with

$$\hat{Q} = \begin{bmatrix} \frac{3}{5} & 0 \\ \frac{4}{5} & 0 \\ 0 & 1 \end{bmatrix}, \quad \hat{R} = \begin{bmatrix} 5 & -10 \\ 0 & 1 \end{bmatrix}$$

and

$$\hat{Q}^*b = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Then by solving

$$Rx = Q^*b,$$

we have

$$x = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

□