Real Analysis 25-09-19

§ 1.5 Convergence of measurable functions.

- $f_n \to f$ a.e. (pointwise convergence)

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{a.e.}$$

- $f_n \to f$ uniformly on $Y \subset X$. (uniform convergence)

$\forall \varepsilon > 0$, $\exists$ $N = N(\varepsilon)$ such that

$$|f_n(y) - f(y)| < \varepsilon \quad \text{for all } n \geq N \text{ and } y \in Y.$$

We write $\boxed{f_n \rightrightarrows f \text{ on } Y.}$

- $f_n \to f$ in L^1.

$$\int_X |f_n - f| \, d\mu \to 0 \quad \text{as } n \to \infty.$$

- $f_n \to f$ in measure.

$\forall \ \rho > 0$, $\quad \mu\{x : |f_n(x) - f(x)| > \rho\} \to 0$ as $n \to \infty$.

Let $(X, \mathcal{M}, \mu)$ be a measure space.

Thm 1.16 (Egorov Thm).

Let $f, f_n, n \geq 1$, be measurable functions, finite a.e.

Suppose $\mu(X) < \infty$, and

$$f_n \to f \quad \text{a.e} \quad \text{as } n \to \infty.$$

Then for any $\varepsilon > 0$, $\exists\ A \in \mathcal{M}$ with $\mu(A) < \varepsilon$ such that

$$f_n \rightrightarrows f \quad \text{on } X \setminus A.$$

Pf. Let $\varepsilon > 0$. WLOG, assume that f, f_n are finite everywhere.

For any $i, j \in \mathbb{N}$, we define

$$A_j^i = \bigcup_{k \geq j} \left\{ x : \ |f_k(x) - f(x)| \geq \frac{1}{2^i} \right\}.$$

Clearly as j increases, A_j^i is monotone decreases.

So $\quad A_j^i \searrow \bigcap_{j=1}^{\infty} A_j^i$

Notice that $x \in \bigcap_{j=1}^{\infty} A_j^i$ implies $\exists$ infinitely many

k such that $|f_k(x) - f(x)| \geq \frac{1}{2^i}$.

So $f_k(x) \not\to f(x)$. Hence $\mu\left(\bigcap_{j=1}^{\infty} A_j^i\right) = 0$.

Since $\mu(X) < \infty$, $A_j^i \searrow \bigcap_{j=1}^{\infty} A_j^i$,

by the continuity of μ, we have

$$\mu(A_j^i) \to \mu\left(\bigcap_{j=1}^{\infty} A_j^i\right) = 0 \quad \text{as } j \to \infty.$$

So for each $i \in \mathbb{N}$, we can find $j(i) \in \mathbb{N}$

such that

$$\mu(A_{j(i)}^i) < \frac{\varepsilon}{2^i}.$$

Now let

$$A = \bigcup_{i=1}^{\infty} A_{j(i)}^i.$$

Then $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_{j(i)}^i) < \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon$.

We claim that

$$f_k \rightrightarrows f \quad \text{on } X \setminus A.$$

To see this, notice $X \setminus A = \bigcap_{i=1}^{\infty} \left(A^i_{j(i)}\right)^c$

Hence $x \in X \setminus A \iff x \notin A^i_{j(i)}$ for all i

But $x \notin A^i_{j(i)} = \bigcup_{k \geq j(i)} \left\{ y : |f_k(y) - f(y)| \geq \frac{1}{2^i} \right\}$

means $|f_k(x) - f(x)| < \frac{1}{2^i}$ for all $k \geq j(i)$

Hence $f_k \rightrightarrows f$ unif on $X \setminus A$.

$\blacksquare$

Remark: The assumption that $\mu(X) < \infty$ can not be dropped in Egorov's thm.

Here is a counter-example.

- Let $X = [0, \infty)$, $\mu = \mathcal{L}_{[0,\infty)}$.

Let $f_k = \chi_{[k,k+1)}$, $k = 1, 2, \ldots$

Then $f_k \to 0$ on $[0, \infty)$

However $\nexists$ no set Y with $\mu(Y) = \infty$

such that $f_k \rightrightarrows 0$ on Y.

To see this

Suppose $f_k \rightrightarrows 0$ on Y.

Then $\exists$ N such

(*) $\left| f_k(y) - 0 \right| < \frac{1}{2}$ for all $y \in Y$ and $k \geq N$

That means $Y \cap [N, \infty) = \emptyset$.

Otherwise if $y \in Y \cap [N, \infty)$,

then $\exists$ $k \geq N$, s.t. $y \in [k, k+1)$, then

$$f_k(y) = \chi_{[k,k+1)}(y) = 1$$

leading to a contradiction with (*).

Hence $Y \subset [0, N)$ and $\mu(Y) < \infty$.

Prop 1.17. If $f_k \to f$ in measure, then $\exists$ a subsequence (k_j) of natural numbers such that

$$\lim_{j\to\infty} f_{k_j}(x) = f(x) \quad \text{a.e.}$$

Pf. Since $f_k \to f$ in measure, so we can find a subsequence $(k_j)_{j=1}^{\infty}$ of natural numbers such that

$$k_{j+1} > k_j$$

and

$$\mu\left\{x : \left|f_{k_j}(x) - f(x)\right| > \frac{1}{j}\right\} \leq \frac{1}{2^j},$$

$$j = 1, 2, \ldots.$$

In what follows, we will prove $f_{k_j} \to f$ a.e.

Define $B_n = \bigcup_{j \geq n} \{ x : |f_{k_j}(x) - f(x)| > \frac{1}{j} \}$.

Then

$$\mu(B_n) \leq \sum_{j \geq n} \mu \{ x : |f_{k_j}(x) - f(x)| > \frac{1}{j} \}$$

$$\leq \sum_{j \geq n} 2^{-j} = 2^{-(n-1)}.$$

Hence $\lim_{n \to \infty} \mu(B_n) = 0$.

In particular $\mu\left(\bigcap_{n=1}^{\infty} B_n \right) = 0$.

Now we claim

$$f_{k_j}(x) \to f(x) \quad \text{for } x \notin \bigcap_{n=1}^{\infty} B_n$$

Indeed if $x \notin \bigcap_{n=1}^{\infty} B_n$, then $x \in \bigcup_{n=1}^{\infty} B_n^c$

So $x \in B_n^c$ for some n.

But $B_n^c = \bigcap_{j \geq n} \{ y : |f_{k_j}(y) - f(y)| < \frac{1}{j} \}$

$x \in B_n^c \Rightarrow$ for any $j \geq n$,

$$|f_{k_j}(x) - f(x)| < \frac{1}{j}$$

$$\Rightarrow \quad f_{k_j}(x) \to f(x).$$

$\boxed{/}$

Prop 1.18. Assume $\mu(X) < \infty$.

Assume f, f_n, $n \geq 1$, are measurable, finite a.e, and

$$f_n \to f \text{ a.e.}$$

Then $f_n \to f$ in measure.

Pf. Let $\rho, \varepsilon > 0$.

We want to show

(**) $\mu\{ x : |f_k(x) - f(x)| > \rho \} < \varepsilon$ when k is large enough.

To see (**), by Egorov's Thm,

we can find $A \in \mathcal{M}$ with $\mu(A) < \varepsilon$

such that

$$f_k \rightrightarrows f \quad \text{on } X \setminus A.$$

Therefore $\exists$ N such that

$$|f_k(x) - f(x)| < \rho \quad \text{for all } x \in X \setminus A$$
$$\text{and } k \geq N$$

So

$$\{x : |f_k(x) - f(x)| \geq \rho\} \subseteq A \quad \text{if } k \geq N$$

Therefore

$$\mu\{x : |f_k(x) - f(x)| \geq \rho\} \leq \mu(A) < \varepsilon$$
$$\text{for all } k \geq N$$

This proves (**).

$\blacksquare$

Remark: In prop. 1.18, the assumption

$$\mu(X) < \infty$$

Can not be dropped.

The same counter-example

$$f_k = \chi_{[k,k+1)}, \quad k \geq 1.$$

$$X = [0, \infty), \quad \mu = \mathcal{L}_{[0,\infty)}.$$

Then $f_k \to 0$. But

$$\bar{\mu}\left\{ |f_k(x) - 0| > \tfrac{1}{2} \right\} = 1 \not\to 0 \text{ as } k \to \infty.$$

Prop. 1.19. Assume $f_k \to f$ in L^1

Then $f_k \to f$ in measure.

Pf. Using Markov inequality, for given $\rho > 0$

$$\mu\{x : |f_k(x) - f(x)| \geq \rho\} \leq \frac{1}{\rho} \cdot \int_X |f_k - f| \, d\mu$$

$$\to 0$$

$\blacksquare$

Corollary 1.20. Assume $f_k \to f$ in L^1. Then $f_{k_j} \to f$ a.e for some subsequence (k_j).

Pf. It is the direct consequence of Prop 1.18, 1.19. $\square$

Chap 2. Outer measures.

There are two ways to construct measure spaces.
One is the approach by Carathéodory via Outer measures,
the other one is by Riesz representation thm on Linear functions.

§ 2.1 Outer measures.

Def. Let $X \neq \emptyset$. An outer measure μ on X is a function from $\mathcal{P}_X = \{Y : Y \subset X\}$ to $[0, +\infty]$ such that

(1) $\mu(\emptyset) = 0$;

(2) If $A \subseteq \bigcup_{j=1}^{\infty} A_j$, then

$$\mu(A) \leq \sum_{j=1}^{\infty} \mu(A_j).$$

(Countable sub-additivity).

Example 1. A well known example of outer measure is the Lebesgue measure $\mathcal{L}$ on $\mathbb{R}$, defined by

$$\lambda(A) = \inf\left\{ \sum_{j=1}^{\infty} |I_j| : A \subset \bigcup_{j=1}^{\infty} I_j , \right.$$
$$\left. I_j = [a_j, b_j] \right\},$$

where $|I_j| = b_j - a_j$.

- Clearly $\lambda(\phi) = 0$ because $\phi \subset [0, \varepsilon]$.
- If $A \subset \bigcup_{j=1}^{\infty} A_j$, then

(*) $\lambda(A) \leq \sum_{j=1}^{\infty} \lambda(A_j)$.

To prove (*), we may $\sum_{j=1}^{\infty} \lambda(A_j) < \infty$; otherwise, there is nothing to prove.

Let $\varepsilon > 0$. By definition, for each j, we can find $\{I_{j,n}\}_{n=1}^{\infty}$ such that $\bigcup_{n=1}^{\infty} I_{j,n} \supset A_j$ and

$$\sum_{n=1}^{\infty} |I_{j,n}| < \lambda(A_j) + \frac{\varepsilon}{2^j}.$$

Then $\bigcup_{j=1}^{\infty} \bigcup_{n=1}^{\infty} I_{j,n} \supseteq \bigcup_{j=1}^{\infty} A_j \supseteq A$

But

$$\underbrace{\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} |I_{j,n}|} < \sum_{j=1}^{\infty} \left(\lambda(A_j) + \frac{\varepsilon}{2^j} \right)$$

$$= \left(\sum_{j=1}^{\infty} \lambda(A_j) \right) + \varepsilon .$$

So $\lambda(A) \leq \sum_{j=1}^{\infty} \lambda(A_j) + \varepsilon .$

Letting $\varepsilon \to 0$ gives $(*)$

Example 2 : Consider $(\mathcal{G}, \varphi)$, where

$$\mathcal{G} \subset \mathcal{P}_X ,$$

and $\varphi : \mathcal{G} \to [0, +\infty)$ which satisfies

(i) $\inf_{G \in \mathcal{G}} \varphi(G) = 0$

(ii) $\bigcup_{j=1}^{\infty} G_j = X$ for some $G_j \in \mathcal{G}$.

Then define for $A \subset X$,

$$\mu(A) = \inf \left\{ \sum_{j=1}^{\infty} \varphi(H_j) : A \subset \bigcup_{j=1}^{\infty} H_j , \ H_j \in \mathcal{G} \right\}$$

Then μ is an outer measure.

(In the case of Leb. on $\mathbb{R}$,

choose $(\mathcal{G}, \varphi)$ by $\mathcal{G} = \{ [a,b] : a < b \}$

$\varphi : \mathcal{G} \to [0, \infty]$ by

$\varphi([a,b]) = b - a$.)

Def. Let μ be an outer measure on X.

We say $E \subset X$ is μ-measurable or simply measurable if

$$(**) \quad \mu(C) = \mu(C \cap E) + \mu(C \cap E^c), \quad \forall C \subset X.$$

Let $\mathcal{M}_C$ denote the collection of all μ-measurable sets.

Thm 2.1 (Carathéodory)

$\mathcal{M}_c$ is a σ-algebra on X.

Moreover, $(X, \mathcal{M}_c, \mu)$ is a measure space.

($E \in \mathcal{M}_c \Leftrightarrow \mu(C) = \mu(C \cap E) + \mu(C \cap E^c), \ \forall C \subset X$).

Pf. First, it is direct to see that $\phi, X \in \mathcal{M}_c$.

Next we show that if $E, F \in \mathcal{M}_c$ then $E \cup F \in \mathcal{M}_c$.

Indeed for any $C \subset X$,

$$\mu(C) = \mu(C \cap E) + \mu(C \cap E^c) \quad (\text{since } E \in \mathcal{M}_c)$$

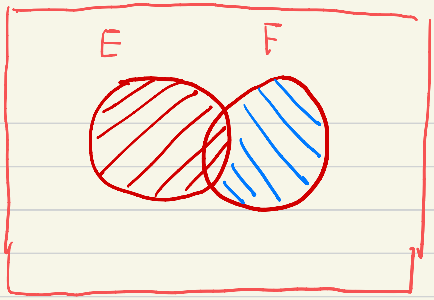
$$= \mu(C \cap E) + \mu(C \cap E^c \cap F) + \mu(C \cap E^c \cap F^c)$$

$$\geq \mu(C \cap (E \cup F)) + \mu(C \cap (E \cup F)^c).$$

Here we use the fact that

$$(C \cap E) \cup (C \cap E^c \cap F) = C \cap (E \cup F)$$

$$(\quad E \cup (E^c \cap F) = E \cup F \ )$$



But by sub-additivity

$$\mu(C) \leq \mu(C \cap (E \cup F)) + \mu(C \cap (E \cup F)^c)$$

Hence we have the "=" and so

$$E \cup F \in \mathcal{M}^c.$$

By Induction, we see that

$$\bigcup_{i=1}^{n} E_i \in \mathcal{M}_c \quad \text{if} \quad E_1, \cdots, E_n \in \mathcal{M}_c.$$

Also, $E_1 \cap E_2 = (E_1^c \cup E_2^c)^c$

So $E_1 \cap E_2 \in \mathcal{M}_c$ if $E_1, E_2 \in \mathcal{M}_c$

Next we prove that if $E_1, E_2, \cdots,$ are disjoint

$$E_i \in M_c$$

then
$$\bigcup_{i=1}^{\infty} E_i \in M_c .$$

To see this, set

$$A_n = \bigcup_{i=1}^{n} E_i ,$$

$$A_\infty = \bigcup_{i=1}^{\infty} E_i .$$

Then $A_n \in M_c$.

Notice that for any $C \subset X$,

$$\mu(C \cap A_n) = \mu(C \cap A_n \cap E_n) + \mu(C \cap A_n \cap E_n^c)$$

(since $E_n \in M_c$)

$$= \mu(C \cap E_n) + \mu(C \cap A_{n-1})$$

$$\overset{\text{Repeating it}}{=} \mu(C \cap E_n) + \mu(C \cap E_{n-1}) + \mu(C \cap A_{n-2})$$

$$= \cdots = \sum_{i=1}^{n} \mu(C \cap E_i)$$

Now

$$\mu(C) = \mu(C \cap A_n) + \mu(C \cap A_n^c)$$

(since $A_n \in \mathcal{M}_C$)

$$= \sum_{i=1}^{n} \mu(C \cap E_i) + \mu(C \cap A_n^c)$$

$$\geqslant \sum_{i=1}^{n} \mu(C \cap E_i) + \mu(C \cap A_\infty^c).$$

Taking $n \to \infty$ gives

$$(***) \quad \mu(C) \geqslant \sum_{i=1}^{\infty} \mu(C \cap E_i) + \mu(C \cap A_\infty^c)$$

$$\geqslant \mu(C \cap A_\infty) + \mu(C \cap A_\infty^c)$$

Hence $A_\infty \in \mathcal{M}_C$.

In the inequality $(***)$, letting $C = A_\infty$

gives $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \geqslant \sum_{i=1}^{\infty} \mu(E_i)$

and so $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$

This proves the countable additivity of μ.

Now let $E_i \in \mathcal{M}_c$, $i \geq 1$, which might not be disjoint. Then let

$$F_1 = E_1$$
$$F_2 = E_2 \setminus E_1$$
$$\cdots$$
$$F_n = E_n \setminus (E_1 \cup \cdots \cup E_{n-1})$$
$$\cdots$$

Then $F_n \in \mathcal{M}_c$, F_n are disjoint,

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n \in \mathcal{M}_c$$

$\blacksquare$

Def: A measure space $(X, \mathcal{M}, \mu)$ is said to be complete if

$A \in \mathcal{M}$ with $\mu(A)=0 \Longrightarrow B \in \mathcal{M}$ for all $B \subset A$.

(A is called a null set)

Prop 2.2. The measure space $(X, \mathcal{M}_c, \mu)$ constructed from an outer measure μ on X is complete.

Pf. Let $A \in \mathcal{M}_c$ with $\mu(A)=0$.

Suppose $B \subset A$.

Then $\mu(B) \leq \mu(A) = 0$.

Hence for any $C \subset X$,

$$\mu(C \cap B) \leq \mu(B) \leq 0, \text{ which implies } \mu(C \cap B) = 0.$$

So

$$\mu(C) \geq \mu(C \setminus B) = \mu(C \setminus B) + \mu(C \cap B)$$

and $\mu(C) = \mu(C \setminus B) + \mu(C \cap B)$. Hence $B \in \mathcal{M}_c$. ▨