Math 6082 2025-03-28

§5.3 Variational principle between topological and measure theoretic entropies.

<u>Thm</u> (Goodwyn, 1968)

Let (X, T) be a TDS. Then

$$h(T) = \sup\{ h_\mu(T) : \mu \in M(X,T) \}.$$

We proved the direction $h(T) \geq h_\mu(T)$ for all $\mu \in M(X,T)$ in the last lecture. Today we will prove the other direction.

We first give some equivalent definitions for $h(T)$.

Def. (Bowen metric and Bowen balls).

Let $T: X \to X$ be a cts map on a compact metric space (X,d).

For $n \in \mathbb{N}$, define a metric d_n on X by

$$d_n(x, y) = \max_{0 \leq i \leq n-1} d(T^i x, T^i y).$$

We can d_n the n-th Bowen metric.

For $\varepsilon > 0$ and $x \in X$, define

$$B_n(x,\varepsilon) = \{ y \in X : d_n(x,y) < \varepsilon \}$$

$$= \bigcap_{i=0}^{n-1} T^{-i} B(T^i x, \varepsilon).$$

We call $B_n(x,\varepsilon)$ an (n,ε)-Bowen ball centered at x.

Def. A subset F of X is said to be (n,ε)-spanning w.r.t. T if $\forall x \in X$, $\exists y \in F$ s.t. $d_n(x,y) \le \varepsilon$, i.e.

$$X \subset \bigcup_{x \in F} \overline{B_n(x,\varepsilon)}.$$

Def. A subset $E \subset X$ is said to be (n,ε)-separated if $d_n(x,y) > \varepsilon$ for all $x, y \in E$ with $x \neq y$.

Def. For $\varepsilon > 0$, $n \in \mathbb{N}$, we define

- $r_n(\varepsilon)$ = the smallest cardinality of any spanning set of X

- $S_n(\varepsilon)$ = the largest cardinality of any (n,ε)-separated subset of X.

<u>Lem 1</u>. $r_n(\varepsilon) \leq S_n(\varepsilon) \leq r_n(\varepsilon/2)$.

Pf. Let E be an (n,ε)-separated set of maximal cardinality. Then E is also an (n,ε)-spanning set for X. If not, then $\exists\ y \in X$ s.t.

$$d_n(x,y) > \varepsilon \text{ for all } x \in E.$$

Then $E \cup \{y\}$ is a new (n,ε)-separated set with larger cardinality, leading to a contradiction.

Hence $r_n(\varepsilon) \leq S_n(\varepsilon)$.

Next we show that $S_n(\varepsilon) \leq r_n(\varepsilon/2)$.

Let E be an (n,ε)-separated set and F an $(n,\varepsilon/2)$ spanning set for X. Then $\forall\ x \in E$, $\exists$ some point $\varphi(x) \in F$ s.t. $d_n(x, \varphi(x)) \leq \varepsilon/2$.

Then $\varphi: E \to F$ is injective, which implies that

$$\# F \geq \# E$$

Hence $\quad s_n(\varepsilon) \leq r_n(\varepsilon/2)$.

Prop 2. (1) If α is an open cover of X with Lebesgue number δ, then

$$N\Big(\bigvee_{i=0}^{n-1} T^{-i}\alpha \Big) \leq r_n(\delta/2) \leq s_n(\delta/2).$$

(2) If $\varepsilon > 0$ and γ is an open cover with $\operatorname{diam}(\gamma) \leq \varepsilon$, then

$$r_n(\varepsilon) \leq s_n(\varepsilon) \leq N\Big(\bigvee_{i=0}^{n-1} T^{-i}\gamma \Big).$$

Pf. (1) By Lem 1, it suffices to show

$$N\Big(\bigvee_{i=0}^{n-1} T^{-i}\alpha \Big) \leq r_n(\delta/2).$$

Let F be an $(n, \delta/2)$-spanning set for X of cardinality $r_n(\delta/2)$.

Then $\quad X = \bigcup_{x \in F} \overline{B_n(x, \delta/2)} := \bigcup_{x \in F} \bigcap_{i=0}^{n-1} T^{-i} \overline{B(T^i x, \delta/2)}$

Since each ball $\overline{B(T^i x, \delta/2)}$ is a set of a member of α, we have

$$N\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) \leq \# F = r_n(\delta/2).$$

(2). Let E be an (n,ε)-separated set of Cardinality $S_n(\varepsilon)$. Since γ is an open cover of diameter $\mathrm{diam}(\gamma) < \varepsilon$, each member of

$$\bigvee_{i=0}^{n-1} T^{-i}\gamma$$

Contains at most 1 point in E.

Hence $N\left(\bigvee_{i=0}^{n-1} T^{-i}\gamma\right) \geq S_n(\varepsilon) \geq r_n(\varepsilon)$.

$\blacksquare$

Cor 4 Let $\varepsilon > 0$. Let α_ε be an open cover of X by all balls of radius 2ε, and β_ε an open cover of X by Balls of radius $\varepsilon/2$.

Then $N\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha_\varepsilon\right) \leq r_n(\varepsilon) \leq S_n(\varepsilon)$

$$\leq N\left(\bigvee_{i=0}^{n-1} T^{-i}\beta_\varepsilon\right).$$

Cor. 5

$$h(T) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n}\log r_n(\varepsilon)$$

$$= \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n}\log S_n(\varepsilon).$$

(these formulae hold with liminf replaced by limsup).

Here, we use

$$h(T) = \lim_{\varepsilon\to 0} h(T, \alpha_\varepsilon) = \lim_{\varepsilon\to 0} h(T, \beta_\varepsilon).$$

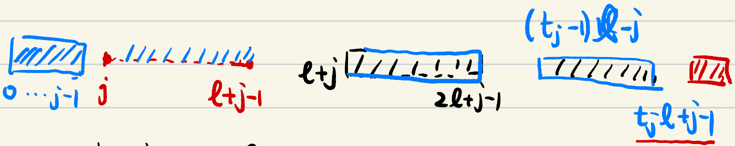
Lem 6. Let $\nu \in \mathcal{M}(X)$. Suppose $\xi = \{A_1, \cdots, A_k\}$ is a partition of $(X, \mathcal{B}(X))$. Then for any integers n, ℓ with $n \geq 2\ell$, we have

$$\frac{1}{n} H_\nu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right) \leq \frac{1}{\ell} H_{\nu_n}\left(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\right) + \frac{2\ell}{n}\log k,$$

when $\nu_n = \frac{1}{n}\left(\nu + \nu\circ T^{-1} + \cdots + \nu\circ T^{-(n-1)}\right)$.

Pf. For any $0 \leq j \leq \ell-1$, let t_j be the largest integer t s.t $t\ell + j \leq n$. Then

$$\bigvee_{i=0}^{n-1} T^{-i}\xi = \bigvee_{r=0}^{t_j-1} T^{-r\ell-j}\left(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\right) \vee \bigvee_{m\in S_j} T^{-m}\xi$$



where S_j is a subset of $\{0, 1, \cdots, n-1\}$ of cardinality $\leq 2\ell$.

Hence

$$H_\nu\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big) \leq \sum_{r=0}^{t_j-1} H_\nu\Big(T^{-r\ell-j}\Big(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\Big)\Big)$$

$$+ 2\ell \log k .$$

Summing over j from 0 to $\ell-1$ gives

$$\ell H_\nu\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big) \leq \sum_{j=0}^{\ell-1}\sum_{r=0}^{t_j-1} H_\nu\Big(T^{-r\ell-j}\Big(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\Big)\Big) + 2\ell^2\log k$$

$$= \sum_{p=0}^{n-\ell} H_\nu\Big(T^{-p}\Big(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\Big)\Big) + 2\ell^2\log k$$

$$= \sum_{p=0}^{n-\ell} H_{\nu\circ T^{-p}}\Big(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\Big) + 2\ell^2\log k$$

$$\leq \sum_{p=0}^{n-1} H_{\nu\circ T^{-p}}\Big(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\Big) + 2\ell^2\log k$$

$$\leq n H_{\nu_n}\Big(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\Big) + 2\ell^2\log k$$

where we use the concavity of $\phi(x) = -x\log x$ in the last inequality.

Hence

$$\frac{1}{n} H_\nu\Big(\bigvee_{i=0}^{n-1} T^{-i}\xi\Big) \leq \frac{1}{\ell} H_{\nu_n}\Big(\bigvee_{i=0}^{\ell-1} T^{-i}\xi\Big) + 2\ell\log k .$$

Pf of the inequality

$$h(T) \le \sup\{ h_\mu(T) : \mu \in M(X,T) \}.$$

Let $\varepsilon > 0$. We will prove that $\exists\ \mu \in M(X,T)$ s.t.

$$\liminf_{n\to\infty} \frac{1}{n} \log S_n(\varepsilon) \le h_\mu(T).$$

For $n \in \mathbb{N}$, let E_n be an (n,ε)-separated set for X with cardinality $S_n(\varepsilon)$. Let

$$\sigma_n = \frac{1}{S_n(\varepsilon)} \sum_{x \in E_n} \delta_x$$

Set

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \sigma_n \circ T^{-i}.$$

Choose a subsequence (n_j) of natural numbers such that

- $\mu_{n_j} \to \mu \in M(X,T)$
- $\frac{1}{n_j} \log S_{n_j}(\varepsilon) \to \liminf_{n\to\infty} \frac{1}{n} \log S_n(\varepsilon).$

Choose a partition $\xi = \{A_1, \cdots, A_k\}$ of $(X, \mathcal{B}(X))$ with $\operatorname{diam} \xi < \varepsilon$ and $\mu(\partial A_i) = 0$ for all $1 \le i \le k$.

Since no member of $\bigvee_{i=0}^{n-1} T^{-i}\xi$ contains more than 1 point of E_n,

$$H_{\sigma_n}\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right) = \log S_n(\varepsilon).$$

By lem 6, for each l,

$$\frac{1}{n_j} \log S_{n_j}(\varepsilon) = \frac{1}{n_j} H_{\sigma_{n_j}}\left(\bigvee_{i=0}^{n_j-1} T^{-i}\xi\right)$$

$$\le \frac{1}{l} H_{\mu_{n_j}}\left(\bigvee_{i=0}^{l-1} T^{-i}\xi\right) + \frac{2l}{n_j} \log k$$

$$\to \frac{1}{l} H_{\mu}\left(\bigvee_{i=0}^{l-1} T^{-i}\xi\right) \quad \text{as } j \to \infty$$

Hence

$$\liminf_{n\to\infty} \frac{1}{n} \log S_n(\varepsilon) \le \frac{1}{l} H_\mu\left(\bigvee_{i=0}^{l-1} T^{-i}\xi\right)$$

$$\to h_\mu(T, \xi). \quad \blacksquare$$