Math 6082 2025-03-14

§5. Topological entropy and topological pressure.

§5.1 Topological entropy.

Alder, Konheim and Mc Andrew (1965) introduced topological entropy as an invariant of topological conjugacy and an analogue of measure theoretic entropy.

Def. Let X be a compact metric space. Let α, β be two open covers of X. set

$$\alpha \vee \beta = \{ U \cap V : U \in \alpha, V \in \beta \}.$$

We call $\alpha \vee \beta$ the join of α and β.

Similarly, we define

$$\vee_{i=1}^{n} \alpha_i$$

the join of any finite collection of open covers of X.

Def. For two open covers α, β of X, write

$$\alpha < \beta$$

if every member in β is a subset of a member in α. We say that β is a refinement of α.

Clearly, $\alpha < \alpha \vee \beta$.

Def If α is an open cover of X, let $N(\alpha)$ denote the number of sets in a finite subcover of α with smallest cardinality.
We define

$$H(\alpha) = \log N(\alpha).$$

Remark: (1) $H(\alpha) \geq 0$

(2) $H(\alpha) \leq H(\beta)$ if $\alpha < \beta$

(3) $H(\alpha \vee \beta) \leq H(\alpha) + H(\beta)$.

(4) If $T: X \to X$ is cts, then

$$H(T^{-1}\alpha) \leq H(\alpha).$$

If T is surjective, then, $H(T^{-1}\alpha) = H(\alpha)$.

(It is possible that $H(T^{-1}\alpha) < H(\alpha)$

e.g. $T: [0,1] \to [0,1]$ by $Tx \equiv 0$.

$\alpha = \{[0, \frac{2}{3}), (\frac{1}{3}, 1]\}$.)

Thm 5.1. If α is an open cover of X and $T: X \to X$ is cts, then

$$\lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \right)$$

exists.

Pf. Write

$$a_n = H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \right).$$

Then

$$a_{n+m} = H\left(\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \right) \vee T^{-n} \left(\bigvee_{i=0}^{m-1} T^{-i}\alpha \right) \right)$$

$$\leq a_n + a_m.$$

$\blacksquare$

Def. If α is an open cover of X and $T: X \to X$ is cts, set

$$h(T, \alpha) := \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha \right)$$

and we call it the entropy of T relative to α

Def. (topological entropy).

Let $T: X \to X$ be cts. The topological entropy of X is defined by $h(T) = \sup_{\alpha} h(T, \alpha)$, where α ranges over all open covers of X.

Thm 5.2. If X_1, X_2 are compact metric spaces, and $T_i : X_i \to X_i$ are cts, and if $\pi : X_1 \to X_2$ is cts and surjective s.t the following diagram commutes,

$$\begin{tikzcd} X_1 \arrow[r, "T_1"] \arrow[d, "\pi"'] & X_1 \arrow[d, "\pi"] \\ X_2 \arrow[r, "T_2"] & X_2 \end{tikzcd}$$

then $h(T_1) \geq h(T_2)$. If π is a homeomorphism, then $h(T_1) = h(T_2)$.

Remark. The above result shows that topological entropy is an invariant of topological conjugacy.

Pf of Thm 5.2.

Let α be an open cover of X_2. Then

$$h(T_2, \alpha) = \lim_{n\to\infty} \frac{1}{n} H\Big(\bigvee_{i=0}^{n-1} T_2^{-i} \alpha \Big)$$

$$= \lim_{n\to\infty} \frac{1}{n} H\Big(\pi^{-1}\Big(\bigvee_{i=0}^{n-1} T_2^{-i} \alpha \Big)\Big)$$ (here we use that π is surjective).

$$= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \pi^{-1} T_2^{-i} \alpha\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} T_1^{-i} \pi^{-1} \alpha\right)$$

$$= h(T_1, \pi^{-1}\alpha).$$

Hence $h(T_1) \geq h(T_2)$. $\blacksquare$

§ 5.2 Calculation of Topological entropy.

Thm 5.3. Let $T: X \to X$ be cts and α is an open cover of X. Suppose that $\operatorname{diam} \bigvee_{i=0}^{n-1} T^{-i}\alpha \to 0$ as $n \to \infty$. Then

$$h(T) = h(T, \alpha).$$

Pf. Let β be an open cover of X, and let δ be a Lebesgue number of β (i.e. $\forall\ A \subset X$ with $\operatorname{diam} A < \delta$, $\exists\ B \in \beta$ s.t $A \subset B$)

Take n s.t $\operatorname{diam} \bigvee_{i=0}^{n-1} T^{-i}\alpha < \delta$.

Then

$$\beta < \bigvee_{i=0}^{n-1} T^{-i}\alpha.$$

Thus
$$H\Big(\bigvee_{j=0}^{m-1} T^{-j}\beta\Big) \leq H\Big(\bigvee_{j=0}^{m-1} T^{-j}\Big(\bigvee_{i=0}^{n-1} T^{-i}\alpha\Big)\Big)$$
$$= H\Big(\bigvee_{i=0}^{n+m-2} T^{-i}\alpha\Big)$$

Hence
$$h(T,\beta) \leq \lim_{m\to\infty} \frac{1}{m} H\Big(\bigvee_{i=0}^{n+m-2} T^{-i}\alpha\Big)$$
$$= h(T,\alpha).$$

$\blacksquare$

Examples: Topolgical entropy of fullshifts & Markov shifts.