Math 6082

2025-01-17

Review.

- Poincaré recurrence Thm.

Let $T: (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure preserving map. Let $B \in \mathcal{B}$ with $\mu(B) > 0$. Then for μ-a.e $x \in B$,

$$T^n x \in B \quad \text{for infinitely many } n \geq 1.$$

- Birkhoff recurrence Thm.

Let (X, T) be a topological dynamical system, i.e.

$T: X \to X$ is a continuous map on a compact metric space X.

Then $\exists\ x \in X$ such that $\exists\ n_i \to \infty$ with

$$T^{n_i} x \to x.$$

Def. such point x is called recurrent.

2.3. Minimality

Def. A TDS (X,T) is called to be minimal, if

$$\{T^n x : n \geq 1\} \text{ is dense in } X$$

for all $x \in X$.

A simple fact: (X,T) is minimal $\Leftrightarrow$ there exists no proper T-invariant subset.

Thm 2.3. Let (X,T) be a TDS. There exists a closed subset Y of X such that $TY = Y$ and (Y,T) is minimal.

Pf. Construct $\mathcal{F}$ as in the proof of Thm 2.1. More details

Let $Y_0 \in \mathcal{F}$ be the minimal element of $\mathcal{F}$.

Notice that $TY_0 \subset Y_0$. We claim that $TY_0 = Y_0$.

Since if $TY_0 \subsetneqq Y_0$, then TY_0 is a proper T-inv subset of Y_0. $\blacksquare$

Example 2.4. Let $\alpha \in (0,1)$ be an irrational number. Let $T: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be the rotation map

$$x \mapsto x + \alpha \pmod 1.$$

Then (X, T) is minimal.

- Due to the fact that $\{n\alpha \pmod 1\}_{n \geq 1}$ is dense in $\mathbb{R}/\mathbb{Z}$.

§ 2.4 Factors and extensions.

Def. A TDS (Y, S) is called a factor of (X, T) if there exists $\pi: X \to Y$ which is continuous and onto such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ Y \arrow[r, "S"] & Y \end{tikzcd}$$

i.e. $\quad \pi \circ T = S \circ \pi$.

In this case, (X, T) is called an extension of (Y, S).

Def. (Kronecker system) Let K be a compact group and $a \in K$. Define $T: K \to K$ by

$$Tx = ax \quad \text{(left multiplication)}$$

We call (K, T) a Kronecker system.

Thm 2.5. Every point in a Kronecker system is recurrent.

This was proved in the last class.

Below is a simple property.

Prop 2.6 Let (Y, S) be a factor of (X, T), and $\pi: X \to Y$ is the factor map. If $x \in X$ is recurrent for (X, T), then πx is recurrent for (Y, S).

Pf. Suppose $x \in X$ is recurrent for (X, T), then

$$\exists\, n_i \nearrow \infty \text{ s.t. } \quad T^{n_i} x \to x$$

So $\quad \pi \circ T^{n_i} x \to \pi x$.

But $\quad \pi \circ T^{n_i} = S^{n_i} \circ \pi$

$$\Rightarrow \quad S^{n_i} \pi x \to \pi x.$$

$\blacksquare$.

We show below that if y_0 is recurrent for (Y, S), then any preimage of y_0 is recurrent, if (X, T) is a group extension.

Def: Let (Y, S) be a TDS and K is a compact group.

Let $\psi: Y \to K$ be a continuous map.

Set $X = Y \times K$ and $T: Y \times K \to Y \times K$ by

$$T(y, k) = (Sy, \psi(y) \cdot k).$$

Then (X, T) is an extension of (Y, S), which is called a group extension of (Y, S).

Thm 2.7. Let y_0 be a recurrent point of (Y, S) and (X, T) be a group extension of (Y, S), $X = Y \times K$. Then (y_0, k) is recurrent for (X, T) for all $k \in K$.

Pf. For $k_1 \in K$, let $R_{k_1} : X \to X$ be defined by

$$R_{k_1}(y, k) = (y, k k_1).$$

Then R_{k_1} commutes with T.

Indeed

$$R_{k_1} T(y, k) = R_{k_1}(Sy, \psi(y) k) = (Sy, \psi(y) k k_1).$$

$$T R_{k_1}(y, k) = T(y, k k_1) = (Sy, \psi(y) k k_1).$$

Let e be the identity of K. We first show that (y_0, e) is recurrent.

For $x \in X$, let $Q(x) = \overline{\{T^n x : n \geq 1\}}$.

Then $TQ(x) \subset Q(x)$, and

$$x \text{ is recurrent} \iff x \in Q(x).$$

Since y_0 is recurrent,

$Q(y_0, e)$ contains (y_0, k_1) for some $k_1 \in K$.

(Notice

$$T(y_0, e) = (sy_0, \psi(y_0)e)$$

$$T^2(y_0, e) = T(sy_0, \psi(y_0)e)$$

$$= (s^2 y_0, \psi(sy_0)\psi(y_0)e)$$

$$\cdots$$

$$T^n(y_0, e) = (s^n y_0, \psi(s^{n-1} y_0)\psi(s^{n-2} y_0) \cdots \psi(y_0) \cdot e).$$

Since R_{k_1} commutes with T, we have

$$R_{k_1} Q(x) = Q(R_{k_1}(x)).$$

Hence

$$\begin{aligned}(y_0, k_1^2) &= R_{k_1}(y_0, k_1) \\ &\in R_{k_1} Q(y_0, e) \\ &= Q(y_0, k_1) \\ &\subset Q(y_0, e)\end{aligned}$$

explain

Similarly,

$$(y_0, k_1), (y_0, k_1^2), \cdots, (y_0, k_1^n) \in Q(y_0, e).$$

However, since $\exists\ n_i \to \infty$ s.t $k_1^{n_i} \to e$,

we obtain that $(y_0, e) \in Q(y_0, e)$.

Then $(y_0, k) = R_k(y_0, e) \in Q(R_k(y_0, e))$

$$= Q(y_0, k).$$

Therefore, (y_0, k) is recurrent.

$\boxed{\checkmark}$

Let us give an application in number theory.

- Let $\alpha \in (0,1)$. Define $T : \mathbb{T}^2 \to \mathbb{T}^2$ by

$$T(x,y) = (x+\alpha,\ y+2x+\alpha)$$

Then $(\mathbb{T}^2, T)$ is a group extension of

$(\mathbb{T}^1, S)$ given by $x \mapsto x+\alpha$.

By Thm 2.7, every point of $\mathbb{T}^2$ is recurrent.

Consider the orbit of $(0,0)$:

$$(0,0) \xrightarrow{T} (\alpha, \alpha) \xrightarrow{T} (2\alpha, 4\alpha) \xrightarrow{T} (3\alpha, 9\alpha)$$

$$\to \cdots \to (n\alpha, n^2\alpha) \to \cdots$$

- As a consequence, $\forall \varepsilon > 0$, $\exists$ $n, m \in \mathbb{Z}$ s.t.

$$|n^2\alpha - m| < \varepsilon.$$

We can extend the above proposition to polynomials of higher degree.

Let $P(x)$ be a polynomial of degree d.

set

$$P_d(x) = P(x),$$

$$P_{d-1}(x) = P_d(x+1) - P_d(x),$$

$$P_{d-2}(x) = P_{d-1}(x+1) - P_{d-1}(x),$$

$$\cdots$$

$$P_0(x) = P_1(x+1) - P_1(x),$$

where $P_0(x) =$ constant α

Consider $T: \mathbb{T}^d \to \mathbb{T}^d$ by

$$(\theta_1, \theta_2, \cdots, \theta_d) \overset{T}{\longmapsto} (\theta_1 + \alpha, \theta_2 + \theta_1, \theta_3 + \theta_2, \cdots, \theta_d + \theta_{d-1}).$$

We have

$$T(P_1(n), \cdots, P_d(n)) = (P_1(n) + P_0(n), P_2(n) + P_1(n), \cdots, P_d(n) + P_{d-1}(n))$$
$$= (P_1(n+1), P_2(n+1), \cdots, P_d(n+1)).$$

In particular

$$T^n(P_1(0), P_2(0), \cdots, P_d(0)) = (P_1(n), P_2(n), \cdots, P_d(n))$$

Hence $\exists\ n_i \to \infty$ s.t

$$P_d(n_i) \to P_d(0).$$

This leads to the following

Thm 2.8. Let $p(x)$ be a real polynomial with $p(0)=0$. Then for any $\varepsilon>0$, $\exists$ $n\in\mathbb{N}$ s.t

$$|p(n)|<\varepsilon.$$