properties of total differential

If $\begin{cases} \cdot\ f, g: \Omega \to \mathbb{R} \ (\Omega \subseteq \mathbb{R}^n, \text{open}) \text{ are differentiable, and} \\ \cdot\ c \in \mathbb{R} \text{ is a constant.} \end{cases}$

Then
(1) $d(f \pm g) = df \pm dg$,

(2) $d(cf) = c\,df$,

(3) $d(fg) = g\,df + f\,dg$,

(4) $d\left(\frac{f}{g}\right) = \frac{g\,df - f\,dg}{g^2}$ provided $g \neq 0$.

Summary (on differentiation of a real-valued function on $\mathbb{R}^n$) $f: \mathbb{R}^n \to \mathbb{R}$

A. Types of differentiations (Derivatives)

- Directional Derivative:

$$D_{\vec{u}} f(\vec{a}) = \lim_{t \to 0} \frac{f(\vec{a} + t\vec{u}) - f(\vec{a})}{t}$$

- Partial Derivative:

$$\frac{\partial f}{\partial x_i}(\vec{a}) = D_{\vec{e_i}} f(\vec{a}), \quad \vec{e_i} = (0, \cdots, 1, \cdots, 0)$$

- Gradient

$$\vec{\nabla} f(\vec{a}) = \left(\frac{\partial f}{\partial x_1}(\vec{a}), \cdots, \frac{\partial f}{\partial x_n}(\vec{a})\right).$$

- Total Differential

$$df(\vec{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{a})\, dx_i$$

- Higher Derivatives

$$\frac{\partial^{k_1+\cdots+k_n} f}{\partial x_1^{k_1}\cdots x_n^{k_n}}(\vec{a})$$

B. Linear approximation

- $L(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a})\cdot(\vec{x}-\vec{a})$
- $f(\vec{x}) = L(\vec{x}) + \varepsilon(\vec{x})$ error term
- f is differentiable at $\vec{a}$

$$\Leftrightarrow \lim_{\vec{x}\to\vec{a}} \frac{|\varepsilon(\vec{x})|}{\|\vec{x}-\vec{a}\|} = 0.$$

In this case, $df \simeq \nabla f$ (by identifying $dx_i = \Delta x_i$)

C. Relations among various concepts

$$C^{\infty} \Rightarrow \cdots \Rightarrow C^{k+1} \Rightarrow C^{k} \cdots \Rightarrow C^{1} \Rightarrow C^{0}$$

• f is C^1 on an open set containing $\vec{a}$

$$\Downarrow \quad \not\Uparrow$$

f is differentiable at $\vec{a}$

$$\searrow \quad \not\nwarrow$$

f is continuous at $\vec{a}$

$$\not\Downarrow \quad \not\Uparrow$$

$$\Downarrow \quad \not\Uparrow$$

$D_{\vec{u}} f(\vec{a})$ exists $\forall \vec{u} \in \mathbb{R}^n$, $\|\vec{u}\|=1$

$$?\Downarrow \quad \not\Uparrow$$

$\frac{\partial f}{\partial x_i}(\vec{a})$ exists, $\forall i=1,\cdots,n$

Counter examples:

eg1: $f: \mathbb{R} \to \mathbb{R}$

$$f(x)=\begin{cases} x^2 \sin\frac{1}{x} & \text{if } x\neq 0, \\ 0 & \text{if } x=0 \end{cases}$$ Volterra's function

$f'(x)$ is not continuous at $x=0$

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

Similarly $g(x)=x^{2k-2}f(x)$ is k-differentiable but $g^{(k)}$ is <u>not continuous</u> at $x=0$.

eg2: $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2} & \text{if } x^2+y^2 \neq 0, \\ 0 & \text{if } x^2+y^2 = 0. \end{cases}$

$D_{\vec{u}} f(0,0)$ exists, $\forall$ unit vector $\vec{u} = (\cos\theta, \sin\theta) \in \mathbb{R}^2$, but

f is $\underbrace{\text{not continuous}}_{\downarrow}$ at $(0,0)$ (check!)

along the curve $x = y^2$

eg3: $f(x,y) = |x+y|$ is continuous on $\mathbb{R}^2$, but $f_x(0,0)$, $f_y(0,0)$ DNE

eg4: $f(x,y) = \sqrt{|xy|}$, $f_x(0,0)$, $f_y(0,0)$ exist (in fact $= 0$)

but $D_{\vec{u}} f(0,0)$ DNE

eg5: All directional derivative exists and equal, but not continuous at one point

$$f(x,y) = \begin{cases} 1, & \text{if } y = x^2,\ x \neq 0, \\ 0, & \text{else} \end{cases}$$

f is not continuous at $(0,0)$, but all directional derivatives in $(0,0)$ are zero.

Review: Matrix Multiplication

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ be an $m \times n$ matrix

$$= \begin{bmatrix} - & \vec{a_1} & - \\ & \vdots & \\ - & \vec{a_m} & - \end{bmatrix}$$

If $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} | \\ \vec{b} \\ | \end{bmatrix}$ be a $n \times 1$-matrix regarded as a column vector in $\mathbb{R}^n$.

Then matrix multiplication,

$$\underbrace{A}_{m \times n}\underbrace{b}_{n \times 1} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + \cdots + a_{1n}b_n \\ \vdots \\ a_{m1}b_1 + a_{m2}b_2 + \cdots + a_{mn}b_n \end{bmatrix} = \underbrace{\begin{bmatrix} \vec{a_1} \cdot \vec{b} \\ \vdots \\ \vec{a_m} \cdot \vec{b} \end{bmatrix}}_{m \times 1}$$

For

$$\underbrace{a^{\perp}}_{1 \times n}\underbrace{B}_{n \times m} = [a_1, \cdots, a_n] \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_1 b_{11} + a_2 b_{21} + \cdots + a_n b_{n1} \\ a_1 b_{12} + a_2 b_{22} + \cdots + a_n b_{n2} \\ \vdots \\ a_1 b_{1m} + a_2 b_{2m} + \cdots + a_n b_{nm} \end{bmatrix}^{\perp} = \underbrace{[\vec{a} \cdot \vec{b_1}, \cdots, \vec{a} \cdot \vec{b_m}]}_{1 \times m}$$

In general: $(m \times n)$ times

$$\underset{n\times m \;\; m\times k}{AB} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1k} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mk} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1m}b_{m1}, & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} + \cdots + a_{1m}b_{m2} & \cdots \\ \vdots & & \vdots \end{bmatrix}$$

$$= \sum_{j=1}^{m} a_{ij} b_{jk}$$

We use Einstein Convention notation

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}}_{2\times 3} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}_{2\times 3}$$

$a_{11} = 1\times 5 + 2\times 8 = 21$, $a_{12} = 1\times 6 + 2\times 9 = 24$, $a_{13} = 1\times 7 + 2\times 10 = 27$.

$a_{21} = 3\times 5 + 4\times 8 = 47$, $a_{22} = 3\times 6 + 4\times 9 = 54$, $a_{23} = 3\times 7 + 4\times 10 = 61$

$$\begin{array}{r} a_1 \\ a_2 \end{array}\begin{bmatrix} \bigcirc \\ \bigcirc \end{bmatrix} \begin{array}{c} b_1 \;\; b_2 \;\; b_3 \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} \vec{a}_1 \cdot \vec{b}_1, & \vec{a}_1 \cdot \vec{b}_2, & \vec{a}_1 \cdot \vec{b}_3 \\ \vec{a}_2 \cdot \vec{b}_1, & \vec{a}_2 \cdot \vec{b}_2, & \vec{a}_3 \cdot \vec{b}_3 \end{bmatrix}$$

Differentiability of Vector-Valued Functions

$$\vec{F}: \Omega \to \mathbb{R}^m, (\Omega \subset \mathbb{R}^n, \text{ open})$$

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$$

Suppose $\frac{\partial f_i}{\partial x_j}(\vec{a})$ exists for each $i=1,\cdots,m$ & $j=1,\cdots,n$.

$$f_i(\vec{x}) = f_i(\vec{a}) + \underbrace{\nabla f_i(\vec{a})}_{1\times n} \cdot \underbrace{(\vec{x}-\vec{a})}_{n\times 1} + \varepsilon_i(\vec{x})$$

Which means

$$\begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix} = \begin{bmatrix} f_1(\vec{a}) \\ \vdots \\ f_m(\vec{a}) \end{bmatrix} + \underbrace{\sum_{j=1}^{n} \frac{\partial f_i(\vec{a})}{\partial x_j} (\vec{x}_j - \vec{a}_j)}_{\text{Einstein convention notation}} + \varepsilon_i(\vec{x})$$

In the following definitions,

- $\vec{f}: \Omega \to \mathbb{R}^m$ ($\Omega \subseteq \mathbb{R}^n$, open)
- $\vec{f}(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix}$
- $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \Omega$
- $\vec{x} - \vec{a} = \begin{bmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{bmatrix}$

Def: Jacobian Matrix of $\vec{f}$ at $\vec{a}$ is defined to be

$$\nabla \vec{f}(\vec{a}) = \begin{bmatrix} — \nabla f_1(\vec{a}) — \\ \vdots \\ — \nabla f_m(\vec{a}) — \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{a}) & \cdots\cdots & \frac{\partial f_1}{\partial x_n}(\vec{a}) \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}(\vec{a}) & \cdots\cdots & \frac{\partial f_m}{\partial x_n}(\vec{a}) \end{bmatrix}$$

Def: Linearization of $\vec{f}$ at $\vec{a}$ is defined to be

$$\vec{L}(\vec{x}) = \vec{f}(\vec{a}) + Df(\vec{a})(\vec{x} - \vec{a})$$

$$\Rightarrow L_i(\vec{x}) = f_i(\vec{a}) + \underbrace{\frac{\partial f_i(\vec{a})}{\partial x_j}(\vec{x_j} - \vec{a_j})}_{\text{Einstein convention notation}}.$$

Def: $\vec{f}$ is said to differentiable at $\vec{a} \in \Omega$,

if $\begin{cases} \cdot \ \frac{\partial f_i}{\partial x_j}(\vec{a}) \text{ exists } \forall i = 1, \dots, m \ \& \ j = 1, \dots, n \\ \cdot \ \text{Error term of the linear approximation} \\ \quad \vec{\varepsilon}(x) = \vec{f}(\vec{x}) - \vec{L}(\vec{x}) \end{cases}$

Satisfies $\displaystyle\lim_{\vec{x}\to\vec{a}} \frac{\|\vec{\varepsilon}(\vec{x})\|}{\|\vec{x}-\vec{a}\|} = 0$.

Remarks (1) $[D\vec{f}(\vec{a})]_{ij} = \frac{\partial f_i}{\partial x_j}(\vec{a})$

(2) $\vec{f}(\vec{x}) = \vec{f}(\vec{a}) + D\vec{f}(\vec{a})(\vec{x}-\vec{a}) + \vec{\varepsilon}(\vec{x})$

(3) If f is real-valued $(m=1)$, then

$$Df(\vec{a}) = \vec{\nabla} f(\vec{a})$$

(4) $\|\vec{\varepsilon}(\vec{x})\|$ & $\|\vec{x}-\vec{a}\|$ are length in $\mathbb{R}^m$ & $\mathbb{R}^n$ respectively.

(5) $\displaystyle\lim_{\vec{x}\to\vec{a}} \frac{\|\vec{\varepsilon}(\vec{x})\|}{\|\vec{x}-\vec{a}\|} = 0 \Leftrightarrow \lim_{\vec{x}\to\vec{a}} \frac{|\varepsilon_i(\vec{x})|}{\|\vec{x}-\vec{a}\|} = 0 \ \forall i$

Hence

$\vec{f}$ is differentiable at $\vec{a} \Leftrightarrow f_i$ is differentiable at $\vec{a}$, $\forall i = 1, \ldots, m$.

Approximation:

$$\vec{f}(\vec{x}) \approx \vec{T}(\vec{x}) = \vec{f}(\vec{a}) + D\vec{f}(\vec{a})(\vec{x}-\vec{a})$$

$$\Rightarrow \vec{f}(\vec{x}) - \vec{f}(\vec{a}) = \underbrace{D\vec{f}(\vec{a})}_{\text{Jacobian matrix}}(\vec{x}-\vec{a})$$

Notation: $d\vec{f} = D\vec{f}(\vec{a})(\vec{x}-\vec{a})$ approximated change of f (total differential

i.e. $\Delta\vec{f} \approx d\vec{f} \Rightarrow d\vec{f} = D\vec{f}(\vec{a})\,d\vec{x}$

Navier–Stokes system

$$V_t - \Delta V + V \cdot \nabla V - \nabla p = 0$$

$$\Rightarrow (V_i)_t - V_{i,\beta\beta} + V_\alpha V_{i,\alpha} - P_{,i} = 0$$

eg: $\vec{f}(x,y) = ((y+1)\ln x,\ x^2 - \sin y + 1)$

(1) Find $D\vec{f}(1,0)$.

(2) Approximate $\vec{f}(0.9, 0.1)$.

Solution (1) $$D\vec{f}(x,y) = \begin{bmatrix} \frac{\partial}{\partial x}[(y+1)\ln x] & \frac{\partial}{\partial y}[(y+1)\ln x] \\ \frac{\partial}{\partial x}[x^2 - \sin y + 1] & \frac{\partial}{\partial y}[x^2 - \sin y + 1] \end{bmatrix} = \begin{bmatrix} \frac{y+1}{x} & \ln x \\ 2x & -\cos y \end{bmatrix}$$

(2) $$\vec{L}(x,y) = \vec{f}(1,0) + D\vec{f}(1,0) \begin{bmatrix} x-1 \\ y-0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$\Rightarrow \vec{f}(0.9, 0.1) \approx \vec{L}(0.9, 0.1)$$

$$= \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -0.1 \\ 0.1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.1 \\ 1.7 \end{bmatrix}$$