MATH 2010

Tutorial 5

Exercises 13.2

Q39 At what points (x, y, z) in space are the functions continuous?

a. $h(x,y,z) = \ln(z - x^2 - y^2 - 1)$

b. $h(x,y,z) = \dfrac{1}{z - \sqrt{x^2+y^2}}$

a. h is continuous at (x,y,z) if $z - x^2 - y^2 - 1 > 0$.

$$z > x^2 + y^2 + 1.$$

$\therefore$ h is continuous at all (x,y,z) such that $z > x^2+y^2+1$.

b. h is continuous at (x,y,z) if $z - \sqrt{x^2+y^2} \neq 0$

$$z \neq \sqrt{x^2+y^2}$$

$\therefore$ h is continuous at all (x,y,z) such that $z \neq \sqrt{x^2+y^2}$.

Q59 Does knowing that $1 - \dfrac{x^2y^2}{3} < \dfrac{\tan^{-1} xy}{xy} < 1$

tell you anything about $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\tan^{-1} xy}{xy}$?

Give reasons for your answer.

Squeeze Theorem (Sandwich Theorem)

Let $f, g, h : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ be functions of n-variables.

If $\begin{cases} \cdot \ g(\vec{x}) \leq f(\vec{x}) \leq h(\vec{x}) \text{ near } \vec{a} \in \Omega \text{ and} \\ \cdot \ \displaystyle\lim_{\vec{x}\to\vec{a}} g(\vec{x}) = \lim_{\vec{x}\to\vec{a}} h(\vec{x}) = L. \end{cases}$

Then $\displaystyle\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$

$$\lim_{(x,y)\to(0,0)} \left(1 - \frac{x^2y^2}{3}\right) = 1 , \quad \lim_{(x,y)\to(0,0)} (1) = 1$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{\tan^{-1}xy}{xy} = 1$$ by the Squeeze Theorem.

Q71 Define $f(0,0)$ in a way that extends f to be continuous at the origin. $f(x,y) = \ln\left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$.

f is continuous at $(0,0)$ if $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \ln\left(\frac{3x^2 - x^2y^2 + 3y^2}{x^2+y^2}\right)$$

$$= \lim_{(x,y)\to(0,0)} \ln\left(\frac{3(x^2+y^2) - x^2y^2}{x^2+y^2}\right)$$

$$= \lim_{r\to 0} \ln\left(\frac{3r^2 - r^4\cos^2\theta \sin^2\theta}{r^2}\right)$$

$$= \lim_{r\to 0} \ln\left(3 - r^2\cos^2\theta\sin^2\theta\right)$$

$$= \ln 3$$

Polar coordinates:
$x = r\cos\theta$
$y = r\sin\theta$
$r^2 = x^2+y^2$

Define $f(0,0) = \ln 3$

Exercises 13.3

Partial Derivatives

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$ (regarding y as constant)

$$\frac{\partial f}{\partial y}(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$ (regarding x as constant)

Q21 $f(x,y) = \int_x^y g(t)\,dt$ (g continuous for all t)

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\int_x^y g(t)\,dt\right) = \frac{\partial}{\partial x}\left(-\int_y^x g(t)\,dt\right)$$

regarding y as constant

$$= -g(x)$$

Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

constant

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\int_x^y g(t)\,dt\right)$$

regarding x as constant

$$= g(y)$$

Q47 $w = x^2\tan(xy)$

Find all the second-order partial derivatives of the function.

(treat y constant) $\frac{\partial w}{\partial x} = x^2\sec^2(xy)\cdot y + 2x\tan(xy) = x^2y\sec^2(xy) + 2x\tan(xy)$

(treat x constant) $\frac{\partial w}{\partial y} = x^2\sec^2(xy)\cdot x = x^3\sec^2(xy)$

$$\frac{\partial^2 w}{\partial x^2} = y\left(x^2\cdot(2\sec(xy)\sec(xy)\tan(xy)\cdot y) + 2x\sec^2(xy)\right) + 2x\sec^2(xy)\cdot y + 2\tan(xy)$$
$$= 2x^2y^2\sec^2(xy)\tan(xy) + 4xy\sec^2(xy) + 2\tan(xy)$$

$$\frac{\partial^2 w}{\partial y^2} = x^3\, 2\sec(xy)\cdot\sec(xy)\tan(xy)\cdot x$$
$$= 2x^4\sec^2(xy)\tan(xy)$$

$\frac{\partial^2 w}{\partial y\partial x}$ $//$ $2x^3y\sec^2(xy)\tan(xy) + 3x^2\sec^2(xy)$ $//$

$$\frac{\partial^2 w}{\partial x\partial y} = \frac{\partial}{\partial x}(x^3\sec^2(xy)) = x^3\cdot 2\sec(xy)\sec(xy)\tan(xy)\cdot y + 3x^2\sec^2(xy)$$