

Exercises 13.1

Q 27 $f(x, y) = \sin^{-1}(y-x)$

(a) find the function's domain.

Domain = $\{(x, y) : -1 \leq y-x \leq 1\}$

(b) find the function's range.

Range = $\{z : -\frac{\pi}{2} \leq z \leq \frac{\pi}{2}\}$

(c) describe the function's level curves.

$$f(x, y) = C_0$$

$$\sin^{-1}(y-x) = C_0$$

$$y-x = \sin(C_0)$$

$$y-x = C, \quad C = \sin(C_0), \quad -1 \leq C \leq 1$$

level curves are straight lines of the form

$$y-x = C \quad \text{where } -1 \leq C \leq 1.$$

(d) find the boundary of the function's domain.

$$\partial A = \{(x, y) : y-x = 1 \\ \text{or } y-x = -1\}$$

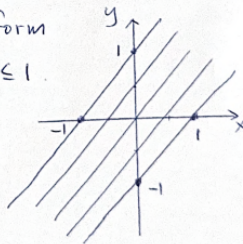
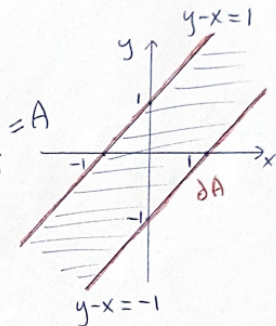
(e) determine whether the domain is an open region, a closed region, or neither.

$$A = \text{Int}(A) \cup \partial A \neq \text{Int}(A)$$

 $\therefore$ Domain A is closed, not open.

(f) decide whether the domain is bounded or unbounded.

Domain is unbounded.



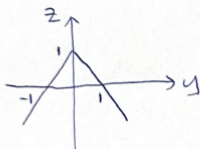
$$S \text{ open} \Leftrightarrow S = \text{Int}(S)$$

$$S \text{ closed} \Leftrightarrow S = \text{Int}(S) \cup \partial S$$

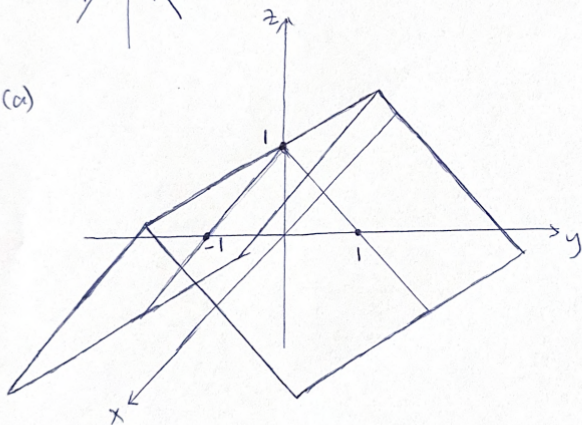
Q45 $f(x,y) = 1 - |y|$

Display the values of the function in two ways :

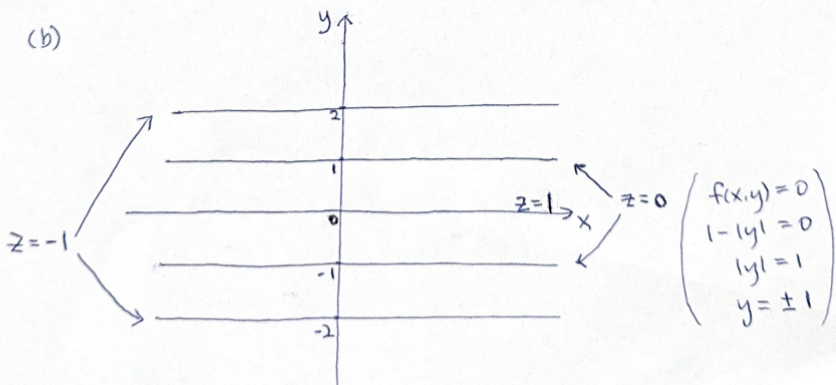
- (a) by sketching the surface $z = f(x,y)$ and
 (b) by drawing an assortment of level curves in the function's domain. Label each level curve with its function value.



(a)



(b)



Exercises 13.2

Q43 $f(x, y) = \frac{x^4 - y^2}{x^4 + y^2}$

By considering different paths of approach, show that the function has no limit as $(x, y) \rightarrow (0, 0)$.

$$\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = L \Leftrightarrow \lim_{\substack{\vec{x} \rightarrow \vec{a} \\ \text{along any path}}} f(\vec{x}) = L$$

$$\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) \text{ does not exist} \Leftrightarrow \lim_{\substack{\vec{x} \rightarrow \vec{a} \\ \text{along path 1}}} f(\vec{x}) \neq \lim_{\substack{\vec{x} \rightarrow \vec{a} \\ \text{along path 2}}} f(\vec{x})$$

$$\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = L \quad \nRightarrow \quad \lim_{\substack{\vec{x} \rightarrow \vec{a} \\ \text{along path 1}}} f(\vec{x}) = \lim_{\substack{\vec{x} \rightarrow \vec{a} \\ \text{along path 2}}} f(\vec{x}) = L$$

$$\lim_{\substack{(x, y) \rightarrow (0, 0) \\ \text{along } y = kx^2}} \frac{x^4 - y^2}{x^4 + y^2} = \lim_{x \rightarrow 0} \frac{x^4 - k^2 x^4}{x^4 + k^2 x^4} = \lim_{x \rightarrow 0} \frac{x^4(1 - k^2)}{x^4(1 + k^2)}$$

$$= \frac{1 - k^2}{1 + k^2} \quad \text{different for different values of } k$$

$\Rightarrow$ different limits along paths $y = kx^2$ of different k .

$\Rightarrow \lim_{(x, y) \rightarrow (0, 0)} \frac{x^4 - y^2}{x^4 + y^2}$ does not exist.

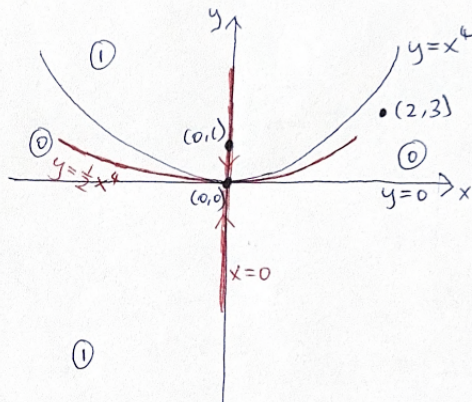
Q55 Let $f(x,y) = \begin{cases} 1, & y \geq x^4 \\ 1, & y \leq 0 \\ 0, & \text{otherwise.} \end{cases}$

Find each of the following limits, or explain that the limit does not exist.

a. $\lim_{(x,y) \rightarrow (0,1)} f(x,y)$

b. $\lim_{(x,y) \rightarrow (2,3)} f(x,y)$

c. $\lim_{(x,y) \rightarrow (0,0)} f(x,y)$



a. When (x,y) near $(0,1)$, it satisfies $y \geq x^4$.

$$f(x,y) = 1 \text{ near } (0,1).$$

$$\lim_{(x,y) \rightarrow (0,1)} f(x,y) = 1$$

b. $f(x,y) = 0$ near $(2,3)$.

$$\lim_{(x,y) \rightarrow (2,3)} f(x,y) = 0$$

c. $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 1$ along $x=0$, $\lim_{(x,y) \rightarrow (0,0)} f(x,y) = 0$ along $y = \frac{1}{2}x^4$

$\Rightarrow \lim_{(x,y) \rightarrow (0,0)} f(x,y)$ does not exist.

Q81 $f(x, y, z) = \frac{x+y+z}{x^2+y^2+z^2+1}$, $\varepsilon = 0.015$

Show that there exists a $\delta > 0$ such that
for all (x, y, z) ,

$$\underbrace{\sqrt{x^2+y^2+z^2}}_{\| (x, y, z) - (0, 0, 0) \|} < \delta \Rightarrow |f(x, y, z) - f(0, 0, 0)| < \varepsilon$$

$$\| (x, y, z) - (0, 0, 0) \|$$

(about continuity at $(0, 0, 0)$)

In general: Let $f: A \rightarrow \mathbb{R}$, $\vec{a} \in A$.

f is continuous at $\vec{a}$ if

$\forall \varepsilon > 0$, $\exists \delta > 0$ such that

if $\vec{x} \in A$ & $\| \vec{x} - \vec{a} \| < \delta$, then $|f(\vec{x}) - f(\vec{a})| < \varepsilon$.

Take $\delta = 0.005$

$$\sqrt{x^2+y^2+z^2} < \delta \Rightarrow |x| < \delta, |y| < \delta, |z| < \delta.$$

$$\Rightarrow |f(x, y, z) - f(0, 0, 0)| = \left| \frac{x+y+z}{x^2+y^2+z^2+1} - 0 \right|$$

$$= \left| \frac{x+y+z}{x^2+y^2+z^2+1} \right| \leq |x+y+z| \quad (\because x^2+y^2+z^2+1 \geq 1)$$

$$\leq |x| + |y| + |z| < 3\delta = 3(0.005) = 0.015 = \varepsilon$$

(For any $\varepsilon > 0$,

take $\delta = \frac{\varepsilon}{3} > 0$.)